STUDY GUIDE OF *DIFFERENTIAL EQUATIONS*

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Abstract. This Study Guide includes the important topics and problems that are featured in the Tests and the Final of *Differential Equations*. Under each topic, examples and exercises from the book by Zill (*A first course in differential equations with modeling applications*, 11th Edition) are listed for more information and practice. You are entitled to a reward of 2 points toward a Test if you are the first person to report a mathematical mistake.

Contents

Topic 0. Classification of differential equations

Given a DE, determine its order and whether it is linear or nonlinear.
(Example 4 and Exercises 1, 3, 5, 7 in §1.1)

Topic 1. First order differential equations

To solve DEs, i.e., equations that involve derivatives, the skills of integration are indispensable, which include the following.

- Integrate polynomials, exponential functions, logarithmic functions, and trigonometric functions, e.g.,
$$\int \frac{1}{x}\,dx = \ln|x| + c \quad \text{and} \quad \int \frac{1}{x^2+1} = \arctan x + c.$$
- Apply substitution.
- Apply integration by parts (IBP).

Moreover, the following skills are also important in DEs.

- Apply trigonometric identities, e.g.,

  $$\sin^2 x + \cos^2 x = 1, \quad \sin(2x) = 2\sin x \cos x, \quad \cos(2x) = 2\cos^2 x - 1 = 1 - 2\sin^2 x.$$

- Apply partial fractions to break a large fraction into smaller ones. For example,

  $$\frac{1}{P - P^2} = \frac{1}{P(1-P)} = \frac{A}{P} + \frac{B}{1-P} = \frac{A(1-P) + BP}{P(1-P)} = \frac{(B-A)P + A}{P(1-P)},$$

  in which A and B are to be determined. Comparing the terms of P and constants in the numerators of both sides, we have that

  $$B - A = 0 \quad \text{and} \quad A = 1.$$

  So $A = 1$ and $B = 1$. Therefore,

  $$\frac{1}{P - P^2} = \frac{1}{P} + \frac{1}{1-P}.$$

Topic 1.A. Autonomous differential equations.

Consider the DE

$$\frac{dy}{dx} = f(y).$$

It is called an autonomous DE because y' depends only on y, i.e., a function of itself.

(1). Find all the equilibrium solutions by solving $f(y) = 0$.
(2). Sketch phase portrait of solutions curves.
(3). Classify the equilibrium solutions as asymptotically stable attractors, semi-stable equilibrium solutions, or unstable repellers.

(Examples 3-5 and Exercises 21, 23, 25, 27 in §2.1)

Problem 1 (Exercise 24 in §2.1)**.** *Let*

$$\frac{dy}{dx} = 10 + 3y - y^2.$$

(a). Find the equilibrium solutions.
(b). Classify each equilibrium solution (asymptotically stable, semi-stable, or unstable).
(b). Sketch the phase portrait.

***Answer*.**

(a). Solving

$$f(y) = 10 + 3y - y^2 = (5 - y)(2 + y) = 0,$$

the equilibrium solutions are $y = 5$ and $y = -2$.

(b). The equilibrium solutions $y = 5$ and $y = -2$ divide the whole xy-plane into three regions.
In the region $\{y > 5\}$, $f(y) < 0$ so $y' = f(y) < 0$ and the function $y(x)$ is decreasing.
In the region $\{-2 < y < 5\}$, $f(y) > 0$ so $y' = f(y) > 0$ and the function $y(x)$ is increasing.
In the region $\{y < -2\}$, $f(y) < 0$ so $y' = f(y) < 0$ and the function $y(x)$ is decreasing.
Therefore, $y = 5$ is an asymptotically stable and $y = -2$ is unstable.

(c). The phase portrait is omitted.

Topic 1.B. Separable differential equations.

Consider the DE

$$\frac{dy}{dx} = g(x)h(y).$$

It is called a separable DE because the variables x, y can be separated to the opposite sides of the DE as follows.

(1). Separate the variables:

$$\frac{dy}{h(y)} = g(x)dx.$$

(2). Integrate both sides:

$$\int \frac{dy}{h(y)} = \int g(x)dx.$$

(3). Simplify the function and the constant to get the general solution.

(4). If the problem is an initial value problem (IVP), that is, there is initial condition $y(x_0) = y_0$, then plug (x_0, y_0) into the general solution to find the constant.

(Examples 3-5 and Exercises 1, 3, 5, 15, 17, 23, 25 in §2.2)

Problem 2 (Exercise 17 in §2.2)**.** *Solve the differential equation*

$$\frac{dP}{dt} = P - P^2.$$

Answer. Separate the variables:

$$\frac{dP}{P - P^2} = dt.$$

Integrate both sides:

$$\int \frac{dP}{P - P^2} = \int dt = t + c_1.$$

Using partial fractions,

$$\int \frac{dP}{P - P^2} = \int \frac{1}{P}\, dP + \int \frac{1}{1 - P}\, dP = \ln|P| - \ln|1 - P| = \ln\left|\frac{P}{1 - P}\right|.$$

Hence,

$$\ln\left|\frac{P}{1 - P}\right| = t + c_1,$$

and

$$\frac{P}{1 - P} = \pm e^{c_1} e^t = ce^t.$$

in which we denote by $c = \pm e^{c_1}$. Then

$$P = ce^t(1 - P) = ce^t - ce^t P,$$

which implies that

$$(1 + ce^t)P = ce^t.$$

So the general solution to the DE is

$$P = \frac{ce^t}{1 + ce^t}.$$

Problem 3 (Exercise 23 in §2.2)**.** *Solve the initial value problem*

$$\frac{dx}{dt} = 4(x^2 + 1), \quad x(\pi/4) = 1.$$

Answer. Separate the variables:

$$\frac{dx}{x^2+1} = 4dt.$$

Integrate both sides:

$$\int \frac{dx}{x^2+1} = \int 4dt.$$

Hence,

$$\arctan(x) = 4t + c,$$

so the general solution to the DE is

$$x = \tan(4t + c).$$

Use the initial condition that $x(\pi/4) = 1$:

$$1 = \tan\left(4 \cdot \frac{\pi}{4} + c\right) = \tan(\pi + c).$$

Hence,

$$\pi + c = \frac{\pi}{4}, \quad \text{and} \quad c = -\frac{3}{4}\pi.$$

The solution to the initial value problem is

$$x = \tan\left(4t - \frac{3}{4}\pi\right).$$

Topic 1.C. Linear differential equations.

Consider the first order linear DE

$$\frac{dy}{dx} + p(x)y = f(x).$$

(1). Multiply both sides of the DE by the integrating factor $e^{\int p}$:

$$e^{\int p(x)}\frac{dy}{dx} + e^{\int p(x)}p(x)y = e^{\int p(x)}f(x).$$

Write the left-hand-side as an integral of a product:

$$\frac{d}{dx}\left(e^{\int p(x)}y\right) = e^{\int p(x)}f(x).$$

(2). Integrate both sides:

$$e^{\int p(x)}y = \int e^{\int p(x)}f(x)\,dx.$$

(3). Simplify the function and the constant.

(4). If the problem is an IVP with initial condition $y(x_0) = y_0$, then plug (x_0, y_0) into the general solution to find the constant.

(Examples 1-5 and Exercises 3, 5, 7, 9, 11, 27, 31 in §2.3)

Problem 4 (Exercise 27 in §2.3)**.** *Solve the initial value problem*

$$xy' + y = e^x, \quad y(1) = 2.$$

Answer.

(1). Divide by x to normalize the DE:

$$y' + \frac{1}{x}y = \frac{e^x}{x}.$$

From this form we identify the integrating factor

$$e^{\int 1/x} = e^{\ln|x|} = e^{\ln x} = x.$$

Here, we take $x \in (0, \infty)$ since the initial condition is at $x = 1$. Multiply by x to the DE and rewrite

$$x\frac{dy}{dx} + y = e^x \quad \text{and} \quad \frac{d}{dx}[xy] = e^x.$$

(2). Integrate both sides:

$$xy = \int e^x\, dx = e^x + c.$$

(3). Solve for y:

$$y = \frac{e^x}{x} + \frac{c}{x}.$$

(4). Use the initial condition that $y(1) = 2$:

$$2 = \frac{e^1}{1} + \frac{c}{1}.$$

Solve for c:

$$c = 2 - e.$$

Hence, the solution to the IVP is

$$y = \frac{e^x}{x} + \frac{2-e}{x}.$$

Topic 1.D. Models by first order differential equations.

The models include

- Newton's law of cooling/warming (Example 4 and Exercises 13, 17 in §3.1),
- mixture of salt solutions (Examples 5-6 and Exercises 21, 27, 29 in §3.1),

Problem 5 (Exercise 23 in §3.1)**.** *A tank contains* 500 *gallons of water and no salt. Water containing* 2 *pounds of salt is pumping into the tank at a rate of* 5 *gallons per minute. Water drains out the tank at the same rate.*

(a). Use differential equation to find the number of pounds of salt $A(t)$ at time t.

(b). Sketch the solution.

Answer.

(a). The input rate of salt is $2 \cdot 5 = 10$ pounds per minute, whereas the output rate of salt is

$$\frac{A}{500} \cdot 5 = \frac{1}{100}A.$$

The DE for the amount of salt in the tank is therefore

$$\frac{dA}{dt} = 10 - \frac{1}{100}A,$$

with initial condition that $A(0) = 0$ since initially there is no salt in the tank.

The above DE is linear and first order. Change it to the standard form:

$$\frac{dA}{dt} + \frac{1}{100}A = 10,$$

Multiply the DE by the integrating factor $e^{\int \frac{1}{100}\,dt} = e^{\frac{1}{100}t}$ and rewrite

$$e^{\frac{1}{100}t}\frac{dA}{dt} + e^{\frac{1}{100}t}\frac{1}{100}A = 10e^{\frac{1}{100}t} \quad \text{and} \quad \frac{d}{dA}\left[e^{\frac{1}{100}t}A\right] = 10e^{\frac{1}{100}t}.$$

Integrate both sides:

$$e^{\frac{1}{100}t}A = \int 10e^{\frac{1}{100}t} = 1000e^{\frac{1}{100}t} + c.$$

Hence,

$$A = 1000 + ce^{-\frac{1}{100}t}.$$

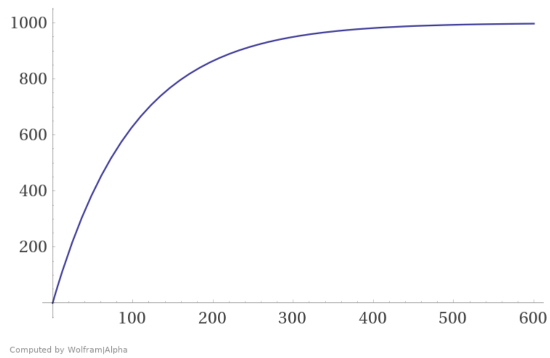
When $t = 0$, $A = 0$:

$$0 = 1000 + c,$$

so $c = -1000$. Thus, the amount of salt in the tank at time t is given by

$$A = 1000 - 1000e^{-\frac{1}{100}t}.$$

(b). The solution curve starts from the origin and converges to $A = 1000$ as $t \to \infty$.



Topic 2. Second order linear differential equations with constant coefficients

Topic 2.A. Homogeneous differential equations.

Consider

$$ay'' + by' + cy = 0, \quad \text{in which } a, b, c \text{ are real numbers.}$$

It is said to be homogeneous because the right-hand-side is zero.

Set $y = e^{mx}$. Then

$$y' = me^{mx} \quad \text{and} \quad y'' = m^2e^{mx}.$$

The DE simplifies to

$$am^2e^{mx} + bme^{mx} + ce^{mx} = e^{mx}\left(am^2 + bm + c\right) = 0.$$

Solve the quadratic equation

$$am^2 + bm + c = 0.$$

Depending on the roots, the general solution to the DE can be represented as follows.

- Case 1. Two distinct real roots m_1 and m_2. Then
$$y = c_1e^{m_1x} + c_2e^{m_2x}.$$
- Case 2. Repeated root, i.e., $m_1 = m_2 = m$. Then
$$y = c_1e^{mx} + c_2xe^{mx}.$$
- Case 3. Two complex and conjugate roots $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, in which $\alpha, \beta \in \mathbb{R}$. Then
$$y = c_1e^{\alpha x}\cos(\beta x) + c_2e^{\alpha x}\sin(\beta x).$$

(Examples 1-4 and Exercises 1, 3, 5, 7, 9, 11, 29, 31 in §4.3)

Topic 2.B. Non-homogeneous differential equations by undetermined coefficients. Consider

$$ay'' + by' + cy = g(x), \quad \text{in which } a, b, c \text{ are real numbers.}$$

It is said to be non-homogeneous because the right-hand-side is non-zero.

(1). Solve the associated homogeneous DE

$$ay'' + by' + cy = 0.$$

The solution $y_c = c_1 y_1 + c_2 y_2$ is called the complimentary solution.

(2). Use undetermined coefficients to find one solution y_p to the non-homogeneous DE. To this end, set y_p according to the non-homogeneous term $g(x)$.
- If $g(x)$ is a polynomial, then set y_p as a polynomial.
- If $g(x)$ is an exponential function, then set y_p to contain the same exponential function.
- If $g(x)$ contains $\cos(kx)$ or $\sin(kx)$, then y_p should contain both $\cos(kx)$ and $\sin(kx)$.

Moreover, compare y_p with the complimentary solution y_c. Make necessary modifications to the setup if y_p is contained in y_c.

(3). The general solution to the non-homogeneous DE is

$$y = y_c + y_p = c_1 y_1 + c_2 y_2 + y_p.$$

(4). If the problem is an initial value problem (IVP), that is, there is initial condition $y(x_0) = y_0$ and $y'(x_0) = y_1$, then plug them into the general solution to find the constants c_1 and c_2.

(Table 4.4.1, Examples 1-11, and Exercises 1, 3, 5, 9, 11, 13, 17, 31 in §4.4)

Problem 6. *Solve the initial value problem by undetermined coefficients.*

$$y'' - y = 2e^{-x} - 2x^2, \quad y(0) = 0, \; y'(0) = 1.$$

Answer.

(1). Solve the associated homogeneous DE

$$y'' - y = 0$$

Set $y = e^{mx}$. Then

$$m^2 - 1 = 0.$$

The two roots are $m = 1$ and $m = -1$. So the complementary solution

$$y_c = c_1 e^x + c_2 e^{-x}.$$

(2). Set

$$y_p = Axe^{-x} + Bx^2 + Cx + D.$$

(Here, the non-homogeneous term contains e^{-x} so we set y_p to contain xe^{-x}, considering that e^{-x} is already contained in the complimentary solution y_c; the non-homogeneous term also contains $-2x^2$ so we set y_p to contain a polynomial of degree two, i.e., $Bx^2 + Cx + D$.)

Compute that

$$y_p' = Ae^{-x} - Axe^{-x} + 2Bx + C,$$

and

$$y_p'' = -Ae^{-x} - Ae^{-x} + Axe^{-x} + 2B = -2Ae^{-x} + Axe^{-x} + 2B.$$

Hence, plug in the DE $y'' - y = 2e^{-x} - 2x^2$:

$$\begin{aligned} y_p'' - y_p &= -2Ae^{-x} + Axe^{-x} + 2B - (Axe^{-x} + Bx^2 + Cx + D) \\ &= -2Ae^{-x} - Bx^2 - Cx + (2B - D) \\ &= 2e^{-x} - 2x^2. \end{aligned}$$

So
$$-2A = 2, \quad -B = -2, \quad C = 0, \quad 2B - D = 0$$
imply that
$$A = -1, \quad B = 2, \quad C = 0, \quad D = 4,$$
and
$$y_p = -xe^{-x} + 2x^2 + 4.$$
(3). The general solution to the non-homogeneous DE is
$$y = y_c + y_p = c_1 e^x + c_2 e^{-x} - xe^{-x} + 2x^2 + 4.$$
(4). Using the initial condition that $y(0) = 0$:
$$c_1 + c_2 + 4 = 0.$$
Using the initial condition that $y'(0) = 1$:
$$y'(x) = c_1 e^x - c_2 e^{-x} - e^{-x} + xe^{-x} + 4x,$$
and
$$y'(0) = c_1 - c_2 - 1 = 1.$$
Solve for c_1 and c_2:
$$c_1 = -1 \quad \text{and} \quad c_2 = -3.$$
Hence, the solution to the IVP is
$$y = -e^x - 3e^{-x} - xe^{-x} + 2x^2 + 4.$$

Topic 2.C. Non-homogeneous differential equations by variation of parameters.
Consider
$$ay'' + by' + cy = g(x), \quad \text{in which } a, b, c \text{ are real numbers.}$$
Normalize the DE:
$$y'' + \frac{b}{a} y' + \frac{c}{a} y = \frac{g(x)}{a}.$$
Denote by $f(x) = g(x)/a$.
(1). Solve the associated homogeneous DE
$$ay'' + by' + cy = 0.$$
The complimentary solution is $y_c = c_1 y_1 + c_2 y_2$.
(2). Set a particular solution
$$y_p = u_1 y_1 + u_2 y_2.$$
Then
$$u_1 = -\int \frac{y_2 f(x)}{W(y_1, y_2)}\, dx \quad \text{and} \quad u_2 = \int \frac{y_1 f(x)}{W(y_1, y_2)}\, dx.$$
Here, $W(y_1, y_2)$ is the Wronskian of y_1 and y_2:
$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'.$$
(3). The general solution to the non-homogeneous DE is
$$y = y_c + y_p = c_1 y_1 + c_2 y_2 + y_p.$$
(4). If the problem is an initial value problem (IVP) with initial condition $y(x_0) = y_0$ and $y'(x_0) = y_1$, then plug them into the general solution to find the constants c_1 and c_2.
(Examples 1-3 and Exercises 1, 5, 11, 13, 19, 27 in §4.6)

Problem 7 (Exercise 27 in §4.6). *The homogeneous equation* $x^2y'' + xy' - \left(x^2 - \frac{1}{4}\right) y = 0$ *has fundamental solution set* $\{x^{-\frac{1}{2}} \cos x, x^{-\frac{1}{2}} \sin x\}$ *on* $(0, \infty)$.
(a). Use variation of parameters to find a particular solution to the non-homogeneous equation

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = x^{\frac{3}{2}}.$$

(b). Write the general solution.

Answer.
(a). Normalize the DE by dividing by x^2:

$$y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right) y = x^{-\frac{1}{2}}.$$

Compute that

$$\begin{aligned}
W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
&= \begin{vmatrix} x^{-\frac{1}{2}} \cos x & x^{-\frac{1}{2}} \sin x \\ -\frac{1}{2}x^{-\frac{3}{2}} \cos x - x^{-\frac{1}{2}} \sin x & -\frac{1}{2}x^{-\frac{3}{2}} \sin x + x^{-\frac{1}{2}} \cos x \end{vmatrix}. \\
&= x^{-\frac{1}{2}} \cos x \left(-\frac{1}{2}x^{-\frac{3}{2}} \sin x + x^{-\frac{1}{2}} \cos x\right) - x^{-\frac{1}{2}} \sin x \left(-\frac{1}{2}x^{-\frac{3}{2}} \cos x - x^{-\frac{1}{2}} \sin x\right) \\
&= x^{-1} \left(\cos^2 x + \sin^2 x\right) \\
&= x^{-1}.
\end{aligned}$$

Set a particular solution

$$y_p = u_1y_1 + u_2y_2.$$

Then

$$u_1 = -\int \frac{x^{-\frac{1}{2}} \sin x \cdot x^{-\frac{1}{2}}}{x^{-1}} dx = -\int \sin x \, dx = \cos x,$$

and

$$u_2 = \int \frac{x^{-\frac{1}{2}} \cos x \cdot x^{-\frac{1}{2}}}{x^{-1}} dx = \int \cos x \, dx = \sin x.$$

Therefore,

$$y_p = u_1y_1 + u_2y_2 = \cos x \cdot x^{-\frac{1}{2}} \cos x + \sin x \cdot x^{-\frac{1}{2}} \sin x = x^{-\frac{1}{2}}(\sin^2 x + \cos^2 x) = x^{-\frac{1}{2}}.$$

(b). The general solution

$$y = c_1x^{-\frac{1}{2}} \cos x + c_2x^{-\frac{1}{2}} \sin x + x^{-\frac{1}{2}}.$$

Topic 2.D. The spring-mass model.

The motion of a mass attached to a spring (which satisfies the Hooke's law) is governed by the following DE.

$$mx'' + \beta x' + kx = f(t).$$

Here, m is the mass (in kilogram), β is the damping constant (in newton-second/meter), k is the spring constant (in newton/meter), and $f(t)$ is the driven force (in newton).

(Examples 1-8 and Exercises 6, 7, 27, 37 in §5.1)

Problem 8. *A mass weighing* 1 *kg is attached to a spring whose spring constant is* 5 *n/m. The damping force equals* 2 *times the instantaneous velocity. Initially, the mass is released from the equilibrium position with a downward velocity of* 2 *m/s.*

(a). Determine the equation of the motion. (Set up the IVP and find the solution.)
(b). Sketch the solution.

Answer.
(a). The IVP reads

$$x'' + 2x' + 5x = 0, \quad x(0) = 0, \; x'(0) = 2.$$

The general solution is

$$x(t) = c_1 e^{-t} \cos(2t) + c_2 e^{-t} \sin(2t).$$

Using the initial condition that $x(0) = 0$:

$$c_1 = 0.$$

Using the initial condition that $x'(0) = 2$:

$$x'(t) = -c_1 e^{-t} \cos(2t) - 2c_1 e^{-t} \sin(2t) - c_2 e^{-x} \sin(2t) + 2c_2 e^{-x} \cos(2t),$$

and

$$x'(0) = -c_1 + 2c_2 = 2.$$

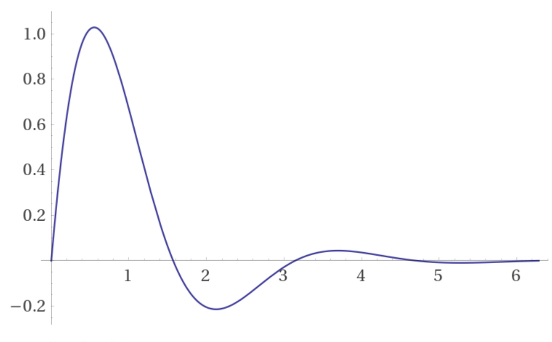
Solve for c_1 and c_2:

$$c_1 = 0 \quad \text{and} \quad c_2 = 1.$$

Hence, the solution to the IVP is

$$x = e^{-t} \sin(2t).$$

(b). The solution curve has magnitude $e^{-t} \to 0$ as $t \to \infty$ and oscillates at a period of π.



Topic 3. The Laplace transform

Topic 3.A. The Laplace transform by integrals.

Let f be a function on $(0, \infty)$. Define the Laplace transform of f as

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\, dt.$$

(Examples 1-6 and Exercises 1, 3, 5, 13, 15, 19, 21, 23, 25, 27, 29, 31 in §7.1)

Some formulas of the Laplace transforms:

- For $n = 0, 1, 2, \ldots,$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0.$$

For example,

$$\mathcal{L}\{1\} = \frac{1}{s} \quad \text{and} \quad \mathcal{L}\{t\} = \frac{1}{s^2}.$$

- $$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a.$$
- $$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}, \quad s > 0.$$
- $$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}, \quad s > 0.$$
- $$\mathcal{L}\{e^{at}f\} = \mathcal{L}\{f\}(s-a).$$

Topic 3.B. Initial value problems by the Laplacian transform.

Write $\mathcal{L}\{y\} = Y(s)$. Then

$$\mathcal{L}\{y'\} = sY(s) - y(0),$$

and

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0).$$

Give an IVP of $y(t)$.

(1). Take Laplace transform on both sides of the DE. The DE then is transformed to an algebraic equations of $Y(s)$. Solve for $Y(s)$ for the algebraic equation.

(2). Take inverse Laplace transform of $Y(s)$ to get $y(t)$ as the solution to the IVP.

(Example 4-5 and Exercises 1, 3, 7, 9, 11, 13, 15, 17, 19, 29, 35, 37, 39, 41 in §7.2)

Problem 9 (Exercise 41 in §7.2)**.** *Consider*

$$y'' + y = \sqrt{2}\sin(\sqrt{2}t), \quad y(0) = 10,\ y'(0) = 0.$$

(a). Take Laplace transform of the above, and solve for $Y(s)$.

(b). Use $Y(s)$ to solve for $y(t)$.

Answer.

(a). Take Laplace transform on both sides of the DE:

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \sqrt{2}\mathcal{L}\{\sin(\sqrt{2}t)\}.$$

Write $Y = \mathcal{L}\{y\}$. Then

$$\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0) = s^2Y(s) - 10s,$$

by the initial conditions that $y(0) = 10$ and $y'(0) = 0$. So the DE becomes

$$s^2Y - 10s + Y = \sqrt{2}\frac{\sqrt{2}}{s^2+(\sqrt{2})^2} = \frac{2}{s^2+2}.$$

Solve for Y in the above algebraic equation

$$Y = \frac{2}{(s^2+1)(s^2+2)} + \frac{10s}{s^2+1}.$$

(b). Take inverse Laplace transform of $Y(s)$ to get $y(t)$ as the solution to the IVP. That is,

$$y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{2}{(s^2+1)(s^2+2)} + \frac{10s}{s^2+1}\right\}.$$

Use partial fractions:

$$\frac{2}{(s^2+1)(s^2+2)} = \frac{A}{s^2+1} + \frac{B}{s^2+2} = \frac{A(s^2+2)+B(s^2+1)}{(s^2+1)(s^2+2)} = \frac{(A+B)s^2+2A+B}{(s^2+1)(s^2+2)}.$$

Hence,

$$A + B = 0 \quad \text{and} \quad 2A + B = 2.$$

Solve for A and B: $A = 2$ and $B = -2$. So

$$\frac{2}{(s^2+1)(s^2+2)} = \frac{2}{s^2+1} + \frac{-2}{s^2+2},$$

and

$$\begin{aligned} y &= \mathcal{L}^{-1}\left\{\frac{2}{(s^2+1)(s^2+2)} + \frac{10s}{s^2+1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{2}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{-2}{s^2+2}\right\} + \mathcal{L}^{-1}\left\{\frac{10s}{s^2+1}\right\} \\ &= 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \sqrt{2}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right\} + 10\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} \\ &= 2\sin(t) - \sqrt{2}\sin(\sqrt{2}t) + 10\cos(t). \end{aligned}$$

Topic 4. First order linear differential systems

Consider

$$\begin{cases} \frac{dx}{dt} = ax + by, \\ \frac{dy}{dt} = cx + dy. \end{cases}$$

Here, a, b, c, d are real numbers. Write the system in matrix form:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Denote

$$\vec{X} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then

$$\vec{X}' = A\vec{X}, \quad \text{in which the matrix } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The eigenvalues are the solutions to the characteristic equation of the matrix A:

$$\det(A - \lambda I) = 0.$$

- Case 1. Two distinct real eigenvalues λ_1 and λ_2 with eigenvectors $\vec{K}_1$ and $\vec{K}_2$, respectively. The eigenvectors are the solutions to

  $$(A - \lambda_1 I)\vec{K}_1 = 0 \quad \text{and} \quad (A - \lambda_2 I)\vec{K}_2 = 0.$$

  Then the general solution

  $$\vec{X} = c_1 e^{\lambda_1 t}\vec{K}_1 + c_2 e^{\lambda_2 t}\vec{K}_2.$$

- Case 2. Repeated eigenvalues λ with eigenvector $\vec{K}$ and generalized eigenvector $\vec{P}$. The generalized eigenvector is the solution to

  $$(A - \lambda I)\vec{P} = \vec{K}.$$

  Then the general solution

  $$\vec{X} = c_1 e^{\lambda t}\vec{K} + c_2\big[te^{\lambda t}\vec{K} + e^{\lambda t}\vec{P}\big].$$

- Case 3. Two complex conjugate eigenvalues $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$. The eigenvalue $\lambda_1 = \alpha + i\beta$ has complex eigenvector
$$\vec{K} = \vec{B}_1 + i\vec{B}_2.$$
in which $\vec{B}_1$ and $\vec{B}_2$ are both real vectors. Then the general solution
$$\vec{X} = c_1 e^{\alpha t}\left[\cos(\beta t)\vec{B}_1 - \sin(\beta t)\vec{B}_2\right] + c_2 e^{\alpha t}\left[\cos(\beta t)\vec{B}_2 + \sin(\beta t)\vec{B}_1\right].$$

(Example 1-6 and Exercises 1, 3, 5, 13, 21, 23, 31, 35, 37, 39, 48 in §8.2)

Depending on the eigenvalues of A, the behaviors of the solution curves are drastically different. The following figure lists all the possible phase portraits.

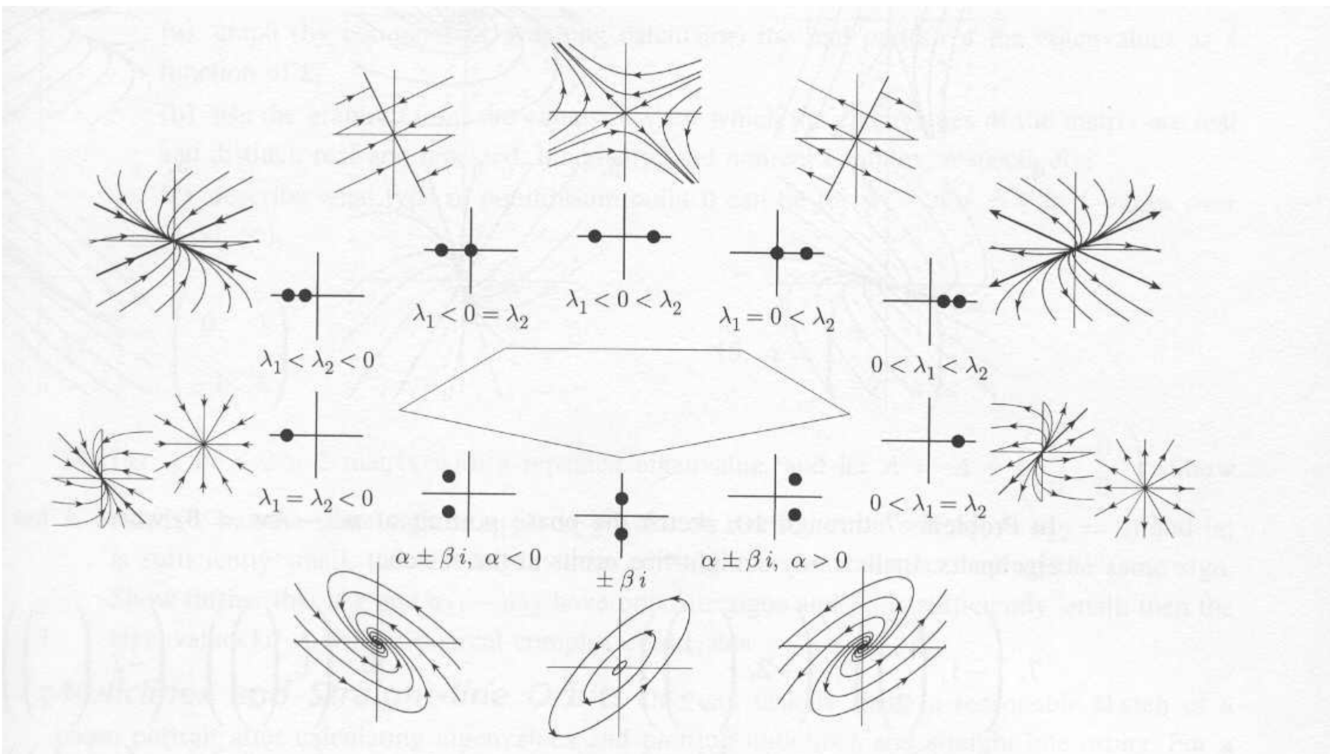


FIGURE 1. Phase portraits of systems of two linear DEs with constant coefficients

Problem 10 (Exercise 1 in §8.2). *Consider*
$$\begin{cases} \frac{dx}{dt} = x + 2y, \\ \frac{dy}{dt} = 4x + 3y. \end{cases}$$

(a). Write as a matrix equation $\vec{X}' = A\vec{X}$.
(b). Find $\det(A - \lambda I)$.
(c). Find the eigenvalues and eigenvectors of A.
(d). Use the above to find the general solution to the system.
(e). Sketch the phase portrait.

Answer.
(a). The matrix equation is
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(b).

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5.$$

(c). The eigenvalues are the solutions to

$$\det(A-\lambda I) = \lambda^2 - 4\lambda - 5 = (\lambda+1)(\lambda-5) = 0.$$

So the two eigenvalues are

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = 5.$$

The eigenvector $\vec{K}_1$ corresponding to the eigenvalue $\lambda_1 = -1$ solves

$$(A - \lambda_1 I)\vec{K}_1 = 0,$$

that is,

$$\begin{pmatrix} 1-\lambda_1 & 2 \\ 4 & 3-\lambda_1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = 0.$$

Hence,

$$\begin{cases} 2p_1 + 2p_2 = 0, \\ 4p_1 + 4p_2 = 0. \end{cases}$$

Set $p_2 = 1$. Then $p_1 = -1$ and the eigenvector

$$\vec{K}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The eigenvector $\vec{K}_2$ corresponding to the eigenvalue $\lambda_2 = 5$ solves

$$(A - \lambda_2 I)\vec{K}_2 = 0,$$

that is,

$$\begin{pmatrix} 1-\lambda_2 & 2 \\ 4 & 3-\lambda_2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = 0.$$

Hence,

$$\begin{cases} -4r_1 + 2r_2 = 0, \\ 4r_1 - 2r_2 = 0. \end{cases}$$

Set $r_1 = 1$. Then $r_2 = 2$ and the eigenvector

$$\vec{K}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(d). The general solution to the system is

$$\vec{X} = c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(e). Refer to the phase portrait when $\lambda_1 < 0 < \lambda_2$ in Figure 1.

Problem 11 (Exercise 31 in §8.2). *Consider*

$$\begin{cases} \frac{dx}{dt} = 2x + 4y, \\ \frac{dy}{dt} = -x + 6y. \end{cases}$$

(a). Write as a matrix equation $\vec{X}' = A\vec{X}$.
(b). Find $\det(A - \lambda I)$.
(c). Find the eigenvalues and eigenvectors of A.
(d). Use the above to find the general solution to the system.

(e). Sketch the phase portrait.

Answer.

(a). The matrix equation is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(b).

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 4 \\ -1 & 6-\lambda \end{vmatrix} = (2-\lambda)(6-\lambda) + 4 = \lambda^2 - 8\lambda + 16.$$

(c). The eigenvalues are the solutions to

$$\det(A - \lambda I) = \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2 = 0.$$

So the repeated eigenvalues are

$$\lambda = 4.$$

The eigenvector $\vec{K}$ corresponding to the eigenvalue $\lambda = 4$ solves

$$(A - \lambda I)\vec{K} = 0,$$

that is,

$$\begin{pmatrix} 2-\lambda & 4 \\ -1 & 6-\lambda \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = 0.$$

Hence,

$$\begin{cases} -2r_1 + 4r_2 = 0, \\ -r_1 + 2r_2 = 0. \end{cases}$$

Set $r_2 = 1$. Then $r_1 = 2$ and the eigenvector

$$\vec{K} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The generalized eigenvector $\vec{P}$ corresponding to the eigenvalue $\lambda = 4$ solves

$$(A - \lambda I)\vec{P} = \vec{K},$$

that is,

$$\begin{pmatrix} 2-\lambda & 4 \\ -1 & 6-\lambda \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \vec{K}.$$

Hence,

$$\begin{cases} -2p_1 + 4p_2 = 2, \\ -p_1 + 2p_2 = 1. \end{cases}$$

Set $p_2 = 1$. Then $p_1 = 1$ and the generalized eigenvector

$$\vec{P} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(d). The general solution to the system is

$$\vec{X} = c_1 e^{4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \left[t e^{4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right].$$

(e). Refer to the phase portrait when $0 < \lambda_1 = \lambda_2$ in Figure 1.

Problem 12 (Exercise 48 in §8.2)**.** *Consider*

$$\begin{cases} \frac{dx}{dt} = 6x - y, \\ \frac{dy}{dt} = 5x + 4y. \end{cases}$$

(a). Write as a matrix equation $\vec{X}' = A\vec{X}$.
(b). Find $\det(A - \lambda I)$.
(c). Find the eigenvalues and eigenvectors of A.
(d). Use the above to find the general solution to the system.
(e). Sketch the phase portrait.

Answer.
(a). The matrix equation is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(b).

$$\det(A - \lambda I) = \begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} = (6-\lambda)(4-\lambda) + 5 = \lambda^2 - 10\lambda + 29.$$

(c). The eigenvalues are the solutions to

$$\det(A - \lambda I) = \lambda^2 - 10\lambda + 29 = (\lambda - 5)^2 + 4 = 0.$$

So the two complex eigenvalues are

$$\lambda_1 = 5 + 2i \quad \text{and} \quad \lambda_2 = 5 - 2i.$$

The eigenvector $\vec{K}$ corresponding to the eigenvalue $\lambda_1 = 5 + 2i$ solves

$$(A - \lambda I)\vec{K} = 0,$$

that is,

$$\begin{pmatrix} 6-\lambda_1 & -1 \\ 5 & 4-\lambda_1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 1-2i & -1 \\ 5 & -1-2i \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = 0.$$

Hence,

$$\begin{cases} (1-2i)r_1 - r_2 = 0, \\ 5r_1 - (1+2i)r_2 = 0. \end{cases}$$

Set $r_1 = 1$. Then $r_2 = 1 - 2i$ and the eigenvector

$$\vec{K} = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

The two real column vectors corresponding to $\lambda_1 = 5 + 2i$ are

$$\vec{B}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{B}_2 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

(d). The general solution to the system is

$$\begin{aligned} \vec{X} &= c_1 e^{5t} \left[\cos(2t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin(2t) \begin{pmatrix} 0 \\ -2 \end{pmatrix} \right] + c_2 e^{5t} \left[\cos(2t) \begin{pmatrix} 0 \\ -2 \end{pmatrix} + \sin(2t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] \\ &= c_1 e^{5t} \begin{pmatrix} \cos(2t) \\ \cos(2t) + 2\sin(2t) \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} \sin(2t) \\ -2\cos(2t) + \sin(2t) \end{pmatrix}. \end{aligned}$$

(e). Refer to the phase portrait when $\alpha \pm \beta i, \alpha > 0$ in Figure 1.