STUDY GUIDE OF *APPLIED DIFFERENTIAL EQUATIONS*

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Abstract. This Study Guide includes the important topics and problems that are featured in the Tests and the Final of *Applied Differential Equations*. Under each topic, examples and exercises from the book by Zill (*A first course in differential equations with modeling applications*, 11th Edition) are listed for more information and practice. You are entitled to a reward of 2 points toward a Test if you are the first person to report a mathematical mistake.

Contents

User Manual:

(I). Make sure you can do all the 19 Problems in this Study Guide **by yourself**.

(II). Under every important topic in each section, I list the Examples and Exercises from the textbook. Read the step-by-step solutions in these Examples provided by the book; then choose an Exercise in the list to practice. (Most of the Exercises listed in this Study Guide have been assigned as homework.) You may choose the odd-numbered exercises so you can compare your solution with the answer key in the back of the book.

Remark. You are to solve a wide variety of differential equations. Always identify the type of the DE[i] and be clear about the method you use. Main types of DEs are given in the following, in which each DE can be incorporated with initial value problems.

- Chapter 2. Autonomous, separable, and linear first order DEs.
- Chapter 3. Linear models by first order linear DEs (e.g. mixture of salt solutions).
- Chapter 4. Homogeneous and nonhomogeneous linear second order DEs.
- Chapter 5. Linear models by second order linear DEs (e.g. spring/mass systems).
- Chapter 7. Initial value problems by the Laplace transform.
- Chapter 8. Systems of two linear first order DEs in the matrix form.

Chapter 1. Introduction to Differential Equations

Important topics in this chapter:

(i). Order and linear vs nonlinear of a DE.

(ii). Verify (explicit, implicit, or integral-defined) solutions of DEs.

(iii). Find interval of solutions of DEs.

1.1. Definitions and Terminology. Give a DE.

- Determine its order and whether it is linear or nonlinear. (Example 4 and Exercises 1-8)
- Verify an explicit solution. (Example 5 and Exercises 11-18)
- Verify an implicit solution. (Example 7 and Exercises 19-20)
- Verify an integral-defined solution. (Exercises 25-28)

1.2. Initial Value Problems. Give an initial value problem (IVP)

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

- Determine the interval of definition of a solution. (Example 2 and Exercises 3-6)
- Determine the region where the IVP has a unique solution:

$$\left\{(x, y) \in \mathbb{R}^2 : f \text{ and } \frac{\partial f}{\partial y} \text{ are continuous}\right\}$$

(Examples 2, 4-5 and Exercises 3-6, 17-28)

Problem 1 (Exercise 24 in §1.2)**.**
Find all (x_0, y_0) such that the initial value problem $(y - x)y' = y + x$, $y(x_0) = y_0$ has a unique solution on a region with x near x_0. Justify your answer. Do not solve.

Answer. The DE can be rewritten as

$$\frac{dy}{dx} = \frac{y + x}{y - x}.$$

[i] Throughout this Guide, DE denotes differential equation.

Inspection of the functions

$$f(x,y) = \frac{y+x}{y-x} \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{2x}{(y-x)^2}$$

shows that they are continuous when $y - x \neq 0$. So through a point (x_0, y_0) in the regions

$$\{y > x\} \quad \text{or} \quad \{y < x\}$$

the differential equation has a unique solution.

Chapter 2. First Order Differential Equations

The important topics in this chapter:

(i). Autonomous DEs:

$$\frac{dy}{dx} = f(y).$$

Find the critical points; sketch the general solutions; classify the critical points.

(ii). Separable DEs:

$$\frac{dy}{dx} = g(x)h(y).$$

Solve the DE by separating independent and dependent variables.

(iii). Linear DEs:

$$\frac{dy}{dx} + p(x)y = f(x).$$

Solve the DE by inserting the integrating factor $e^{\int P(x)}$.

(iv). Use substitution to solve some DEs.

Remark (Review of integrals).

1. Integrate polynomials, rational functions, exponential functions, logarithmic functions, and trigonometric functions, e.g. $\int 1/x = \ln|x| + c$ and $\int 1/(x^2+1) = \arctan x + c$.
2. Use substitution to integrate.
3. Use integration by parts to integrate.
4. Use identities to simplify trigonometric functions, e.g.

$$\sin^2 x + \cos^2 x = 1, \quad \sin(2x) = 2\sin x \cos x, \quad \cos(2x) = 2\cos^2 x - 1 = 1 - 2\sin^2 x.$$

5. Use partial fractions to simplify rational functions, e.g.

$$\frac{1}{P - P^2} = \frac{1}{P(1-P)} = \frac{1}{P} + \frac{1}{1-P}.$$

2.1. Solution Curves Without a Solution. Give a DE

$$\frac{dy}{dx} = f(x,y).$$

- Use the direction field to sketch the solution curves of the DE. (Examples 1-2 and Exercises 1-4)
- Determine whether the DE is autonomous, that is, an autonomous DE has the form

$$\frac{dy}{dx} = f(y).$$

(1). Find all the critical points by solving $f(y) = 0$.

(2). Sketch the general solutions, i.e. draw the phase portrait.

(3). Classify the critical points as asymptotically stable attractors, semi-stable critical points, or unstable repellers.

(Examples 3-5 and Exercises 21-30, 38-40)

Problem 2 (Exercise 24 in §2.1)**.**
Find the critical points and phase portrait of

$$\frac{dy}{dx} = 10 + 3y - y^2.$$

Classify each critical point as asymptotically stable, unstable, or semi-stable. By hand, sketch typical solutions curves in the regions in the xy-plane determined by the graphs of the equilibrium solutions.

Answer. Let

$$f(y) = 10 + 3y - y^2 = (5 - y)(2 + y).$$

The critical points are $y = 5$ and $y = -2$.

In the region above the equilibrium solution $y = 5$, $f(y) < 0$ so $y'(x) = f(y) < 0$ and the function $y(x)$ is decreasing.

In the region below the equilibrium solution $y = 5$ and above the equilibrium solution $y = -2$, $f(y) > 0$ so $y'(x) = f(y) > 0$ and the function $y(x)$ is increasing.

In the region below the equilibrium solution $y = -2$, $f(y) < 0$ so $y'(x) = f(y) < 0$ and the function $y(x)$ is decreasing.

Therefore, the critical point 5 is an asymptotically stable attractor and the critical point -2 is a unstable repeller.

The phase portrait (i.e. general solutions curves) is omitted.

2.2. Separable Equations. Give a DE

$$\frac{dy}{dx} = g(x)h(y).$$

(1). Step 1. Determine if the DE is separable. If yes, then separate! Move all terms containing y to one side and move all terms containing x to the other side:

$$\frac{dy}{h(y)} = g(x)dx.$$

(2). Step 2. Integrate both sides:

$$\int \frac{dy}{h(y)} = \int g(x)dx.$$

(3). Step 3. Simplify the constant.
(4). Step 4. We only do this step if the problem is an initial value problem (IVP). That is, if there is initial condition $y(x_0) = y_0$, then plug (x_0, y_0) into the general solution to find the constant.

(Examples 3-5 and Exercises 1-28)

Problem 3 (Exercise 23 in §2.2)**.**
Find an explicit solution to the initial value problem

$$\frac{dx}{dt} = 4(x^2 + 1), \quad x(\pi/4) = 1.$$

Answer. The DE is separable. Separate the DE:

$$\frac{dx}{x^2 + 1} = 4dt.$$

Integrate both sides:

$$\int \frac{dx}{x^2 + 1} = \int 4dt.$$

Hence,

$$\arctan(x) = 4t + c$$

and so

$$x = \tan(4t + c).$$

Use the initial condition that $x(\pi/4) = 1$:

$$1 = \tan\left(4 \cdot \frac{\pi}{4} + c\right) = \tan(\pi + c).$$

Hence,

$$\pi + c = \frac{\pi}{4}, \quad \text{and} \quad c = -\frac{3}{4}\pi.$$

The solution to the initial value problem is

$$x = \tan\left(4t - \frac{3}{4}\pi\right).$$

2.3. Linear Equations. Give a DE. Determine whether it is linear. If yes, then change it into the standard form

$$\frac{dy}{dx} + p(x)y = f(x).$$

(1). Step 1. Multiply both sides of the DE by the integrating factor $e^{\int P(x)}$:

$$e^{\int P(x)}\frac{dy}{dx} + e^{\int P(x)}p(x)y = e^{\int P(x)}f(x).$$

Write the left-hand-side as an integral of a product:

$$\frac{d}{dx}\left(e^{\int P(x)}y\right) = e^{\int P(x)}f(x).$$

(2). Step 2. Integrate both sides:

$$e^{\int P(x)}y = \int e^{\int P(x)}f(x)\,dx.$$

(3). Step 3. Simplify the constant.

(4). Step 4. We only do this step if the problem is an initial value problem (IVP). That is, if there is initial condition $y(x_0) = y_0$, then plug (x_0, y_0) into the general solution to find the constant.

(Examples 1-5 and Exercises 1-36)

Problem 4 (Exercise 27 in §2.3)**.**
Solve the initial value problem

$$xy' + y = e^x, \quad y(1) = 2.$$

Answer.

(1). Step 1. Dividing by x, the standard form of the given DE is

$$y' + \frac{1}{x}y = \frac{e^x}{x}.$$

From this form we identify $P(x) = 1/x$ and $f(x) = e^x/x$ and further observe that P and f are continuous on $x \in (0, \infty)$. Hence the integrating factor is

$$e^{\int 1/x} = e^{\ln|x|} = e^{\ln x} = x.$$

Multiply by x to the DE and rewrite

$$x\frac{dy}{dx} + y = e^x \quad \text{and} \quad \frac{d}{dx}[xy] = e^x.$$

(2). Step 2. Integrate both sides:

$$xy = \int e^x \, dx = e^x + c.$$

(3). Step 3. Solve for y:

$$y = \frac{e^x}{x} + \frac{c}{x}.$$

(4). Step 4. Use the initial condition that $y(1) = 2$:

$$2 = \frac{e^1}{1} + \frac{c}{1}.$$

Solve for c:

$$c = 2 - e.$$

Hence, the solution to the IVP is

$$y = \frac{e^x}{x} + \frac{2-e}{x}.$$

2.5. Solutions by Substitutions. Give a DE. Use the substitution **provided in the problem** to solve the DE. You **do not** need to remember the substitution rules for different types of DEs!

- In Example 1, use the substitution $y = ux$ to solve
$$(x^2 + y^2)dx + (x^2 - xy)dy = 0.$$
- In Example 2, use the substitution $u = y^{-1}$ to solve
$$x\frac{dy}{dx} + y = x^2y^2.$$
- In Example 3, use the substitution $u = -2x + y$ to solve
$$\frac{dy}{dx} = (-2x + y)^2 - 7, \quad y(0) = 0.$$

(Examples 1-3 and Exercises 1-8, 11-12, 15-19, 21, 23-28)

Chapter 3. Modeling with First Order Differential Equations

The important models include

(i). growth and decay of population, C-14, etc. (Examples 1-3 and Exercises 1-8, 11-12),
(ii). Newton's law of cooling/warming (Example 4 and Exercises 13-18),
(iii). mixture of salt solutions (Examples 5-6 and Exercises 21-25, 27-28),
(iv). series circuits (Example 7 and Exercises 29-30).

3.1. Linear Models. Give a model.

(1). Translate the information in the problem and set up the DE. In this section, it is a first order linear DE.
(2). Solve the DE.

Read the background in the textbook and understand the problems! Notice that the numbers you encounter in Tests and Final are easy to calculate. You should keep the fractions and **do not** simplify them to decimals!

Problem 5 (Exercise 13 in §3.1)**.**
A thermometer with initial temperature $70°$ *is placed outside where the temperature is* $10°$*.*
(a). Solve the differential equation for Newton's law of cooling to derive the temperature of the thermometer $T(t)$*.*
(b). Graph the solution $T(t)$*.*

Answer.

(a). The differential equation for Newton's law of cooling is

$$\frac{dT}{dt} = k(T - T_m),$$

in which T_m is the ambient temperature. Here, $T_m = 10$. So the initial value problem is

$$\frac{dT}{dt} = k(T - 10), \quad T(0) = 70.$$

The above differential equation is separable. Separate the DE:

$$\frac{dT}{T - 10} = kdt.$$

Integrating both sides,

$$\log|T - 10| = kt + c_1,$$

which implies that

$$|T - 10| = e^{kt+c_1} = e^{c_1}e^{kt}.$$

So

$$T = \pm c_1 e^{kt} + 10.$$

Denote $c = \pm e^{c_1}$. Then

$$T = ce^{kt} + 10.$$

When $t = 0$, $T = 70$:

$$70 = c + 10,$$

so $c = 60$. Thus, the temperature of the thermometer

$$T = 60e^{kt} + 10.$$

(b). Graph is omitted here. You can consult Figure 3.1.6 on page 89 in the textbook. Notice that the graph for this problem starts at $T = 70$ when $t = 0$, i.e. the initial condition that $T(0) = 70$, and then converges to the asymptotic $T = 10$ as $t \to \infty$, i.e. the temperature of the thermometer converges to the ambient temperature.

Problem 6 (Exercise 23 in §3.1)**.**
A tank contains 500 *gallons of water and no salt. Water containing* 2 *pounds of salt is pumping into the tank at a rate of* 5 *gallons per minute. Water drains out the tank at the same rate.*
(a). Set up a differential equation for the amount of salt in the tank, $x(t)$.
(b). Solve for $x(t)$.
(c). Graph the solution $x(t)$.

Answer.

(a). The input rate of salt is $2 \cdot 5 = 10$ pounds per minute; the output rate of salt is

$$\frac{x}{500} \cdot 5 = \frac{1}{100}x.$$

The differential equation for the amount of salt in the tank is therefore

$$\frac{dx}{dt} = 10 - \frac{1}{100}x,$$

with initial condition that $x(0) = 0$ (since initially there is no salt in the tank.)

(b). The above differential equation is linear and first order. Change it to the standard form:

$$\frac{dx}{dt} + \frac{1}{100}x = 10,$$

Multiply the DE by the integrating factor $e^{\int \frac{1}{100}} = e^{\frac{1}{100}t}$ and rewrite

$$e^{\frac{1}{100}t}\frac{dx}{dt} + e^{\frac{1}{100}t}\frac{1}{100}x = 10e^{\frac{1}{100}t} \quad \text{and} \quad \frac{d}{dx}\left[e^{\frac{1}{100}t}x\right] = 10e^{\frac{1}{100}t}.$$

Integrate both sides:

$$e^{\frac{1}{100}t}x = \int 10e^{\frac{1}{100}t} = 1000e^{\frac{1}{100}t} + c.$$

Hence,

$$x = 1000 + ce^{-\frac{1}{100}t}.$$

When $t = 0$, $x = 0$:

$$0 = 1000 + c,$$

so $c = -1000$. Thus, the amount of salt in the tank at time t is given by

$$x(t) = 1000 - 1000e^{-\frac{1}{100}t}.$$

(c). Graph is omitted here. You can consult Figure 3.1.6 on page 89 in the textbook. Notice that the graph for this problem starts at the origin since $x(0) = 0$, and then converges to the asymptotic $x = 1000$ as $t \to \infty$.

Problem 7 (Exercise 29 in §3.1)**.**
Suppose an RL circuit has inductance $L = 0.1$ H, resistance $50\ \Omega$, and a constant voltage $E(t) = 30$ volt is applied.
(a). Solve for the currant $\boldsymbol{i}(t)$, if $\boldsymbol{i}(0) = 0$ Amp.
(b). Sketch the current $\boldsymbol{i}(t)$.

Answer.

(a). The linear differential equation for the current $\mathbf{i}(t)$ is

$$L\frac{d\mathbf{i}}{dt} + R\mathbf{i} = E(t).$$

With $L = 0.1$, $R = 50$, and $E(t) = 30$, we have that

$$\frac{1}{10}\frac{d\mathbf{i}}{dt} + 50\mathbf{i} = 30.$$

Multiply both sides by 10:

$$\frac{d\mathbf{i}}{dt} + 500\mathbf{i} = 300.$$

Multiply the DE by the integrating factor $e^{\int 500} = e^{500t}$ and rewrite

$$e^{500t}\frac{d\mathbf{i}}{dt} + e^{500t}500\mathbf{i} = 300e^{500t}.$$

Integrate both sides:

$$e^{500t}\mathbf{i} = \int 300e^{500t}\,dt = \frac{3}{5}e^{500t} + c.$$

Hence,

$$\mathbf{i} = \frac{3}{5} + ce^{-500t}.$$

When $t = 0$, $\mathbf{i} = 0$:

$$0 = \frac{3}{5} + c,$$

so $c = -3/5$. Thus, the current

$$\mathbf{i}(t) = \frac{3}{5} - \frac{3}{5}e^{-500t}.$$

(b). Graph is omitted here. Notice that the graph for this problem starts at the origin since $\mathbf{i}(0) = 0$, and then converges to the asymptotic $\mathbf{i} = 3/5$ as $t \to \infty$.

Chapter 4. Higher Order Differential Equations

The important topics in this chapter:

(i). Give an n-th order linear DE. The general solution is a linear combination of fundamental set of solutions:

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n.$$

Here, we use Wronskian to check that $y_1, ..., y_n$ are linearly independent.

(ii). Reduction of order: Given y_1 as a solution to a second order linear DE, set $y_2 = uy_1$ to solve for y_2.

(iii). Second order homogeneous linear DE with constant coefficients

$$ay'' + by' + cy = 0, \quad \text{where } a, b, c \text{ are numbers.}$$

Use $y = e^{mx}$ to deduce a quadratic equation and solve the DE in three cases: The quadratic equation $am^2 + bm + c = 0$ has two distinct real roots (Case 1), two equal real roots (Case 2), or two conjugate complex roots (Case 3).

(iv). Second order nonhomogeneous linear DE with constant coefficients

$$ay'' + by' + cy = g, \quad \text{where } a, b, c \text{ are numbers.}$$

The general solution $y = y_c + y_p$, in which y_c solves the associated homogeneous DE $ay'' + by' + cy = 0$ and we deduce y_p by methods of Undetermined Coefficients or Variation of Parameters.

(v). Solve systems of first order linear DEs by elimination.

4.1. Preliminary Theory – Linear Equations. Give a linear n-th order DE

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dx}{dx} + a_0(x)y = g(x).$$

- The general solution is a linear combination of fundamental set of solutions:

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n.$$

- There are n equations to define the initial value problem (IVP):

$$y(x_0) = y_0, y'(x_0) = y_1, ..., y^{(n-1)}(x_0) = y_{n-1}.$$

Determine the interval of solutions: The IVP has a unique solution in an interval I where $a_n(x)$, $a_{n-1}(x)$, ..., $a_1(x)$, $a_0(x)$, $g(x)$ are continuous and $a_n(x) \neq 0$.

- Give a homogeneous[i] linear second order DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0.$$

The general solution is a linear combination of two functions.

  - If we find two such functions y_1 and y_2, then we can compute their Wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

[i]Homogeneous means the right-hand-side of the DE equals 0.

  - If the Wronskian $W(y_1, y_2) \neq 0$ on an interval I, then we say the two functions y_1 and y_2 are linearly independent on I and they form the fundamental set of solutions to the DE on I.
  - The general solution to the DE is a linear combination of y_1 and y_2:
$$y = c_1 y_1 + c_2 y_2 \quad \text{on the interval } I.$$
  - If there is initial condition $y(x_0) = y_0$ and $y'(x_0) = y_1$, then use these information to the general solution to find the constants c_1 and c_2. You need to solve a system of linear equations for c_1 and c_2.
- The nonhomogeneous[i] linear second order DE
$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$
has general solution
$$y = y_c + y_p = \text{complementary function} + \text{any particular solution}.$$
Here, y_c solves the homogeneous DE $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ and y_p is a solution to the nonhomogeneous DE.
See §4.4 Undetermined Coefficients and §4.6 Variation of Parameters for finding y_p of second order linear DEs.

(Example 9 and Exercises 1-4, 9-10, 25-26, 28-29)

4.2. Reduction of Order. Give a homogeneous linear second order DE
$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0.$$
We expect the general solution is a linear combination
$$y = c_1 y_1 + c_2 y_2.$$
Provide y_1. Solve for y_2:
(1). normalized the DE to the form
$$y'' + P(x)y' + Q(x)y = 0.$$
(2). use the formula
$$y_2 = y_1 \int \frac{e^{-\int P}}{y_1^2}.$$
(Examples 1-2 and Exercises 9-12)

Problem 8 (Exercise 10 in §4.2)**.**
Suppose that we know that $y_1(x) = x^2$ *solves* $x^2y'' + 2xy' - 6y = 0$ *on* $(0, \infty)$.
(a). Use reduction of order to find a second solution $y_2(x)$ *to* $x^2y'' + 2xy' - 6y = 0$.
(b). Show that $y_1(x)$ *and* $y_2(x)$ *are linearly independent.*
(c). Write the general solution to $x^2y'' + 2xy' - 6y = 0$ *on* $(0, \infty)$.

Answer.
(a). Divide the DE by x^2:
$$y'' + \frac{2}{x}y' - \frac{6}{x^2}y = 0.$$

[i] Nonhomogeneous means the right-hand-side of the DE does not equal 0.

Use reduction of order with $P(x) = 2/x$:

$$\begin{aligned} y_2(x) &= y_1 \int \frac{e^{-\int P}}{y_1^2} \\ &= x^2 \int \frac{e^{-\int 2/x}}{(x^2)^2} \\ &= x^2 \int \frac{e^{-2\ln|x|}}{x^4} \\ &= x^2 \int \frac{x^{-2}}{x^4} \\ &= x^2 \int x^{-6} \\ &= -\frac{1}{5}x^{-3}. \end{aligned}$$

(b). To show that $y_1(x)$ and $y_2(x)$ are linearly independent, we compute the Wronskian:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & -\frac{1}{5}x^{-3} \\ 2x & \frac{3}{5}x^{-4} \end{vmatrix} = x^{-2} \neq 0$$

on $(0, \infty)$.

(c). The general solution to $x^2y'' + 2xy' - 6y = 0$ is

$$y(x) = c_1y_1 + c_3y_2 = c_1x^2 - \frac{c_3}{5}x^{-3} = c_1x^2 + c_2x^{-3},$$

replacing $-c_3/5$ by c_2.

4.3. Homogeneous Linear Equations with Constant Coefficients. Give a second order homogeneous linear DE with constant coefficients

$$ay'' + by' + cy = 0, \quad \text{where } a, b, c \text{ are numbers.}$$

Set $y = e^{mx}$. So

$$y' = me^{mx} \quad \text{and} \quad y'' = m^2e^{mx}.$$

The DE simplifies to

$$am^2e^{mx} + bme^{mx} + ce^{mx} = e^{mx}\left(am^2 + bm + c\right) = 0.$$

Solve the quadratic equation

$$am^2 + bm + c = 0.$$

- Case 1. If there are two distinct real roots m_1 and m_2, then the fundamental set of solutions is

$$e^{m_1x} \quad \text{and} \quad e^{m_2x}.$$

- Case 2. If the two roots are equal, i.e. $m_1 = m_2 = m$, then the fundamental set of solutions is

$$e^{mx} \quad \text{and} \quad xe^{mx}.$$

- Case 3. If there are two complex and conjugate roots $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, then the fundamental set of solutions is

$$e^{\alpha x}\cos(\beta x) \quad \text{and} \quad e^{\alpha x}\sin(\beta x).$$

(Examples 1-4 and Exercises 1-24, 29-32, 49-58)

4.4. Undetermined Coefficients. Give a nonhomogeneous second order linear DE

$$ay'' + by' + cy = g, \quad \text{where } a, b, c \text{ are numbers.}$$

Attention: Use the method of Undetermined Coefficients only for constant coefficients (i.e. a, b, c are numbers) and g contains polynomials, exponential, and cos or sin.

If a, b, c are not numbers or g contains rational, log, tan, $\sin^2$, etc, then use the method of Variation of Parameters in §4.6.

(1). Step 1. Solve the associated homogeneous linear DE

$$ay'' + by' + cy = 0$$

by §4.3. The complimentary solution is $y_c = c_1 y_1 + c_2 y_2$.

(2). Step 2. Use undetermined coefficients to find one solution y_p to the nonhomogeneous DE. To find one solution y_p to the nonhomogeneous DE, depending on the nonhomogenous term $g(x)$, we choose different forms of y_p.
- If g contains polynomial, then y_p should contain polynomial (with possibly higher degree).
- If g contains exponential e^{jx}, then y_p should contain e^{jx}.
- If g contains $\cos(kx)$ or $\sin(kx)$, then y_p should contain both $\cos(kx)$ and $\sin(kx)$.

(3). Step 3. The general solution to the nonhomogeneous DE is

$$y = y_c + y_p = c_1 y_1 + c_2 y_2 + y_p.$$

(4). Step 4. We only do this step if the problem is an initial value problem (IVP). That is, if there is initial condition $y(x_0) = y_0$ and $y'(x_0) = y_1$, then plug them into the general solution to find the constants c_1 and c_2.

(Table 4.4.1, Examples 1-11, and Exercises 1-21, 27-30)

Problem 9.
Solve the initial value problem by undetermined coefficients.

$$y'' - y = 2e^{-x} - 2x^2, \quad y(0) = 0, \; y'(0) = 1.$$

Answer.

(1). Step 1. Solve the associated homogeneous linear DE

$$y'' - y = 0$$

The associated quadratic equation is

$$m^2 - 1 = 0.$$

The two roots are $m = 1$ and $m = -1$. So the complementary solution

$$y_c = c_1 e^x + c_2 e^{-x}.$$

(2). Step 2. Set

$$y_p = Axe^{-x} + Bx^2 + Cx + D.$$

(Here, the nonhomogeneous term contains e^{-x} so we set y_p to contain Axe^{-x}; the nonhomogeneous term also contains $-2x^2$ so we set y_p to contain $Bx^2 + Cx + D$.)

Compute that

$$y_p' = Ae^{-x} - Axe^{-x} + 2Bx + C,$$

and

$$y_p'' = -Ae^{-x} - Ae^{-x} + Axe^{-x} + 2B = -2Ae^{-x} + Axe^{-x} + 2B.$$

Hence, plug in the DE $y'' - y = 2e^{-x} - 2x^2$:

$$\begin{aligned} y_p'' - y_p &= -2Ae^{-x} + Axe^{-x} + 2B - (Axe^{-x} + Bx^2 + Cx + D) \\ &= -2Ae^{-x} - Bx^2 - Cx + (2B - D) \\ &= 2e^{-x} - 2x^2. \end{aligned}$$

So

$$-2A = 2, \quad -B = -2, \quad C = 0, \quad 2B - D = 0$$

imply that

$$A = -1, \quad B = 2, \quad C = 0, \quad D = 4,$$

and

$$y_p = -xe^{-x} + 2x^2 + 4.$$

(3). Step 3. The general solution to the nonhomogeneous DE is

$$y = y_c + y_p = c_1e^x + c_2e^{-x} - xe^{-x} + 2x^2 + 4.$$

(4). Step 4. Using the initial condition that $y(0) = 0$:

$$c_1 + c_2 + 4 = 0,$$

using the initial condition that $y'(0) = 1$:

$$y'(x) = c_1e^x - c_2e^{-x} - e^{-x} + xe^{-x} + 4x,$$

and

$$y'(0) = c_1 - c_2 - 1 = 1.$$

Solve for c_1 and c_2:

$$c_1 = -1 \quad \text{and} \quad c_2 = -3.$$

Hence, the solution is

$$y = -e^x - 3e^{-x} - xe^{-x} + 2x^2 + 4.$$

4.6. Variation of Parameters. Give a nonhomogeneous second order linear DE

$$y'' + P(x)y' + Q(x)y = f(x).$$

If the DE is not in this form, change it.

(1). Step 1. Solve the associated homogeneous linear DE

$$y'' + P(x)y' + Q(x)y = 0.$$

The complimentary solution is $y_c = c_1y_1 + c_2y_2$. You may be provided y_1 and y_2 as in Exercise 27-28.

(2). Step 2. Use variation of parameters to find one solution y_p to the nonhomogeneous DE. That is, set

$$y_p = u_1y_1 + u_2y_2.$$

Then we have that

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0, \\ u_1'y_1' + u_2'y_2' = f(x). \end{cases}$$

Solve for u_1' and u_2'. We can use either of the following.

- Solve the system of equations directly by elimination or substitution.
- Formulas:

$$u_1' = -\frac{y_2f(x)}{W(y_1, y_2)} \quad \text{and} \quad u_2' = \frac{y_1f(x)}{W(y_1, y_2)}.$$

Here, $W(y_1, y_2)$ is the Wronskian of y_1 and y_2.

Attention: Integrate u_1' and u_2' to solve for u_1 and u_2 so y_p is solved.

(3). Step 3. The general solution to the nonhomogeneous DE is

$$y = y_c + y_p = c_1 y_1 + c_2 y_2 + y_p.$$

(4). Step 4. We only do this step if the problem is an initial value problem (IVP). That is, if there is initial condition $y(x_0) = y_0$ and $y'(x_0) = y_1$, then plug them into the general solution to find the constants c_1 and c_2.

(Examples 1-3 and Exercises 1-15, 23-28)

Problem 10 (Exercise 27 in §4.6)**.**
The homogeneous equation $x^2y''+xy'-\left(x^2-\frac{1}{4}\right)y=0$ *has fundamental solution set* $\{x^{-\frac{1}{2}}\cos x, x^{-\frac{1}{2}}\sin x\}$ *on* $(0,\infty)$*.*
(a). Use variation of parameters to find a particular solution to the nonhomogeneous equation

$$x^2y''+xy'+\left(x^2-\frac{1}{4}\right)y=x^{\frac{3}{2}}.$$

(b). Write the general solution.

Answer.
(a). We first set the DE in the standard form by dividing by x^2:

$$y''+\frac{1}{x}y'+\left(1-\frac{1}{4x^2}\right)y=x^{-\frac{1}{2}}.$$

Set the particular solution

$$y_p = u_1y_1 + u_2y_2,$$

in which

$$y_1 = x^{-\frac{1}{2}}\cos x \quad \text{and} \quad y_2 = x^{-\frac{1}{2}}\sin x.$$

Then we have that

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0, \\ u_1'y_1' + u_2'y_2' = f(x), \end{cases}$$

in which $f(x) = x^{-\frac{1}{2}}$,

$$y_1' = -\frac{1}{2}x^{-\frac{3}{2}}\cos x - x^{-\frac{1}{2}}\sin x \quad \text{and} \quad y_2' = -\frac{1}{2}x^{-\frac{3}{2}}\sin x + x^{-\frac{1}{2}}\cos x.$$

The system of equations becomes

$$\begin{cases} u_1'x^{-\frac{1}{2}}\cos x + u_2'x^{-\frac{1}{2}}\sin x = 0, \\ \left(-\frac{1}{2}x^{-\frac{3}{2}}\cos x - x^{-\frac{1}{2}}\sin x\right)u_1' + \left(-\frac{1}{2}x^{-\frac{3}{2}}\sin x + x^{-\frac{1}{2}}\cos x\right)u_2' = x^{-\frac{1}{2}}. \end{cases}$$

Multiply $x^{\frac{1}{2}}$ to the first and second equations:

$$\begin{cases} u_1'\cos x + u_2'\sin x = 0, \\ \left(-\frac{1}{2}x^{-1}\cos x - \sin x\right)u_1' + \left(-\frac{1}{2}x^{-1}\sin x + \cos x\right)u_2' = 1. \end{cases}$$

Solve for u_1' and u_2':

$$u_1' = -\sin x \quad \text{and} \quad u_2' = \cos x.$$

Hence,

$$u_1 = \int -\sin x = \cos x \quad \text{and} \quad u_2 = \int \cos x = \sin x.$$

The particular solution

$$y_p(x) = u_1y_1 + u_2y_2 = \cos x \cdot x^{-\frac{1}{2}}\cos x + \sin x \cdot x^{-\frac{1}{2}}\sin x = x^{-\frac{1}{2}}(\sin^2 x + \cos^2 x) = x^{-\frac{1}{2}}.$$

(b). The general solution
$$y(x) = c_1 x^{-\frac{1}{2}} \cos x + c_2 x^{-\frac{1}{2}} \sin x + x^{-\frac{1}{2}}.$$

Remark. Alternatively, in the above problem, one can use the formulas to derive u_1' and u_2'.

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} x^{-\frac{1}{2}} \cos x & x^{-\frac{1}{2}} \sin x \\ -\frac{1}{2} x^{-\frac{3}{2}} \cos x - x^{-\frac{1}{2}} \sin x & -\frac{1}{2} x^{-\frac{3}{2}} \sin x + x^{-\frac{1}{2}} \cos x \end{vmatrix}. \\ &= x^{-\frac{1}{2}} \cos x \left(-\frac{1}{2} x^{-\frac{3}{2}} \sin x + x^{-\frac{1}{2}} \cos x \right) - x^{-\frac{1}{2}} \sin x \left(-\frac{1}{2} x^{-\frac{3}{2}} \cos x - x^{-\frac{1}{2}} \sin x \right) \\ &= x^{-1} \left(\cos^2 x + \sin^2 x \right) \\ &= x^{-1}. \end{aligned}$$

With $f(x) = x^{-1/2}$, we have that
$$u_1' = -\frac{y_2 f(x)}{W(y_1, y_2)} = -\frac{x^{-\frac{1}{2}} \sin x \cdot x^{-\frac{1}{2}}}{x^{-1}} = -\sin x,$$
and
$$u_2' = \frac{y_1 f(x)}{W(y_1, y_2)} = \frac{x^{-\frac{1}{2}} \cos x \cdot x^{-\frac{1}{2}}}{x^{-1}} = \cos x.$$
One then resume the solution as in above.

4.9. Solving Systems of Differential Equations by Elimination. (Examples 1 and Exercises 1-4, 21-22)

Problem 11 (Exercise 4 in §4.9)**.**
Solve the system of DEs by elimination.
$$\begin{cases} \frac{dx}{dt} - 4y = 1, \\ \frac{dy}{dt} + x = 2. \end{cases}$$

Answer.
(1). Step 1. Rewrite the system to
$$\begin{cases} Dx - 4y = 1, \quad ① \\ Dy + x = x + Dy = 2. \quad ② \end{cases}$$

Here, $D = d/dt$. Notice that x and y are dependent variables and t is the independent variable.
(2). Step 2.
- Eliminate x to get a DE of y: ① $- D \times$ ②.
$$(Dx - 4y) - D(x + Dy) = 1 + D(2).$$
Notice that D on the right-hand side of ② is 0 since it is constant. We have that
$$-4y - D^2 y = -y'' - 4y = 1.$$
Solve the above second order nonhomogeneous linear DE (by methods of Undermined Coefficients or Variation of Parameters):
$$y(t) = y_c(t) + y_p(t) = c_1 \cos(2t) + c_2 \sin(2t) - \frac{1}{4}.$$

- Eliminate y to get a DE of x: $D \times ① + 4 \times ②$.

$$D(Dx - 4y) + 4(x + Dy) = D(1) + 4 \times 2.$$

Notice that D on the right-hand side of ② is 0 since it is constant. We have that

$$D^2 x + 4x = x'' + 4x = 8.$$

Solve the above second order nonhomogeneous linear DE (by methods of Undermined Coefficients or Variation of Parameters):

$$x(t) = x_c(t) + x_p(t) = c_3 \cos(2t) + c_4 \sin(2t) + 2.$$

(3). Step 3. Plug $x(t)$ and $y(t)$ into any equation in the system to reduce the four constants c_1, c_2, c_3, c_4 to two constants. That is, apply

$$x(t) = c_3 \cos(2t) + c_4 \sin(2t) + 2 \quad \text{and} \quad y(t) = c_1 \cos(2t) + c_2 \sin(2t) - \frac{1}{4}$$

to the first equation in the system

$$\frac{dx}{dt} - 4y = 1.$$

We have that

$$\frac{dx}{dt} = -2c_3 \sin(2t) + 2c_4 \cos(2t)$$

and

$$\begin{aligned} \frac{dx}{dt} - 4y &= -2c_3 \sin(2t) + 2c_4 \cos(2t) - 4\left(c_1 \cos(2t) + c_2 \sin(2t) - \frac{1}{4}\right) \\ &= (-2c_3 - 4c_2) \sin(2t) + (2c_4 - 4c_1) \cos(2t) + 1 = 1. \end{aligned}$$

This forces

$$-2c_3 - 4c_2 = 0 \quad \text{and} \quad 2c_4 - 4c_1 = 0.$$

So

$$c_3 = -2c_2 \quad \text{and} \quad c_4 = 2c_1.$$

Hence, the general solution to the system of linear system

$$x(t) = -2c_2 \cos(2t) + 2c_1 \sin(2t) + 2 \quad \text{and} \quad y(t) = c_1 \cos(2t) + c_2 \sin(2t) - \frac{1}{4}.$$

(4). Step 4. We only do this step if the problem is an initial value problem (IVP). That is, if there is initial condition $x(t_0) = x_0$ and $y(t_0) = y_0$, then plug them into the general solution to find the constants c_1 and c_2.

Chapter 5. Modeling with Higher Order Differential Equations

The important topic in this chapter is to use second order linear DE to solve the spring/mass model.

5.1. Linear Models: Initial Value Problems. Give a spring/mass problem. We determine the motion of the mass under the influence of four possible forces:

- Gravitational force.
- Restoring force of the spring.
- Damping force of the environment (e.g. air, fluid, etc).
- Driven/external force.

(1). Translate the information in the problem. Determine one of the following cases depending on the forces in consideration.
  - Free undamped motion, i.e. gravitational force and restoring force are considered. See §5.1.1. (Examples 1 and Exercises 1-6)
  - Free damped motion, i.e. gravitational force, restoring force, and damping force are considered. See §5.1.2. (Examples 3-5 and Exercises 21-31)
  - Driven motion, i.e. gravitational force, restoring force, damping force, and driven force are considered. See §5.1.3. (Examples 6-8 and Exercises 33-37, 41-42)

(2). Set up the DE. Denote $x(t)$ for the position of the mass. So the DE has x as the dependent variable and t as the independent variable.

Let m be the mass, k be the spring constant, β be the damping constant, and $f(t)$ be the driven force. These constants or functions are provided in the problem.
  - Free undamped motion: A second order homogeneous linear DE
$$m\frac{dx^2}{dt^2} + kx = 0.$$
Here, $\omega = \sqrt{k/m}$ is the frequency and $T = 2\pi/\omega$ is the period.
  - Free damped motion: A second order homogeneous linear DE
$$m\frac{dx^2}{dt^2} + kx + \beta\frac{dx}{dt} = 0.$$
Here, there are three cases: overdamped (two distinct real roots), critically damped (two equal roots), and underdamped (two conjugate and complex roots).
  - Driven motion: A second order nonhomogeneous linear DE
$$m\frac{dx^2}{dt^2} + kx + \beta\frac{dx}{dt} = f(t).$$
Here, $x(t) = x_c(t) + x_p(t)$, where $x_c(t)$ is the transient term and $x_p(t)$ is the steady-state term.

(3). Solve the DE with initial conditions $x(t_0) = x_0$ (initial position) and $x'(t_0) = x_1$ (initial velocity). Notice that $x(t)$ is positive below the equilibrium position and is negative above the equilibrium position; $x'(t)$ is positive when moving downward and is negative when moving upward.

Read the background in the textbook and understand the problems! Notice that the numbers you encounter in Tests and Final are in the metric system and easy to calculate. You should keep the fractions and **do not** simplify them to decimals!

Problem 12.

A ball on a spring has mass $m = 2$ kg. Suppose the spring has spring constant $k = 2$, with resistance 4 times the velocity.

(a). Suppose the ball starts at equilibrium with no initial velocity, and a force $f(t) = 2\sin t$ is applied. Set up an initial value problem for the position of the ball.

(b). Solve the initial value problem.

(c). Identify the transient solution.

(d). Identify the steady-state solution.

***Answer*.**

(a). The differential equation reads

$$m\frac{dx^2}{dt^2} = -kx - \beta\frac{dx}{dt} + f(t),$$

in which $m = 2$, $k = 2$, $\beta = 4$, and $f(t) = 2\sin t$. Thus,

$$2\frac{dx^2}{dt^2} = -2x - 4\frac{dx}{dt} + 2\sin t,$$

that is

$$2x'' + 4x' + 2x = 2\sin t.$$

The initial conditions are $x(0) = 0$ and $x'(0) = 0$.

(b). Solve the associated homogeneous linear DE

$$2x'' + 4x' + 2x = 0.$$

The complementary solution

$$x_c = c_1te^{-t} + c_2e^{-t}.$$

Set

$$x_p = A\sin t + B\cos t.$$

(Here, the nonhomogeneous term is $2\sin t$ so we set y_p to contain both $\sin t$ and $\cos t$.)

Compute that

$$x_p' = A\cos t - B\sin t,$$

and

$$x_p'' = -A\sin t - B\cos t.$$

Hence, plug in the DE $2x'' + 4x' + 2x = 2\sin t$:

$$\begin{aligned} 2x_p'' + 4x_p' + 2x_p &= 2(-A\sin t - B\cos t) + 4(A\cos t - B\sin t) + 2(A\sin t + B\cos t) \\ &= (-2A - 4B + 2A)\sin t + (-2B + 4A + 2B)\cos t \\ &= -4B\sin t + 4A\cos t \\ &= 2\sin t. \end{aligned}$$

Therefore, $-4B = 2$ and $4A = 0$. So $B = -1/2$ and $A = 0$. The particular solution

$$x_p = -\frac{1}{2}\cos t.$$

The general solution is

$$x = x_c + c_p = c_1te^{-t} + c_2e^{-t} - \frac{1}{2}\cos t.$$

Using the initial condition that $x(0) = 0$:

$$c_2 - \frac{1}{2} = 0,$$

using the initial condition that $y'(0) = 1$:

$$x'(t) = c_1e^{-t} - c_1te^{-t} - c_2e^{-t} + \frac{1}{2}\sin t,$$

and

$$x'(0) = c_1 - c_2 = 0.$$

Solve for c_1 and c_2:

$$c_1 = \frac{1}{2} \quad \text{and} \quad c_2 = \frac{1}{2}.$$

Hence, the solution is

$$x = \frac{1}{2}te^{-t} + \frac{1}{2}e^{-t} - \frac{1}{2}\cos t.$$

(c). The transient solution is

$$\frac{1}{2}te^{-t} + \frac{1}{2}e^{-t}.$$

(d). The steady-state solution is

$$-\frac{1}{2}\cos t.$$

Chapter 7. The Laplace Transform

The important topics in this chapter:

(i). The Laplace transform of polynomials, exponentials, sin and cos, and piecewise functions.

(ii). The Laplace transform of derivatives and integrals: Let $Y(s) = \mathcal{L}\{y\}$. Then

$$\mathcal{L}\{y'\} = sY(s) - y(0), \quad \mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0),$$

and

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{Y(s)}{s}.$$

(iii). The inverse Laplace transform of rational functions. Partial fractions are needed here.

(iv). Use Laplace transform to solve IVPs. Here, we need to use the Laplace transforms and inverse Laplace transforms of the above functions or formulas provided in the problem. (See Sections 7.2 and 7.4.)

7.1. Definition of the Laplace Transform. Give a function f. Define the Laplace transform of f as

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st}f(t)\,dt.$$

To evaluate the Laplace transform of a function, we need

(1). integrate by parts, if necessary,

(2). notice that $e^{-st} \to 0$ as $t \to \infty$ when $s > 0$,

(3). specify the domain of s.

(Examples 1-4 and Exercises 11-16, 19-32, 37-38)

The important Laplace transforms are

- For $n = 0, 1, 2, ...,$

  $$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0.$$

  For example,

  $$\mathcal{L}\{1\} = \frac{1}{s} \quad \text{and} \quad \mathcal{L}\{t\} = \frac{1}{s^2}.$$

- $$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a.$$

- $$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}, \quad s > 0.$$

- $$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}, \quad s > 0.$$

- Piecewise functions. (Example 6 and Exercises 1-10)

7.2. Inverse Transforms and Transforms of Derivatives. The important inverse Laplace transforms are

- For $n = 0, 1, 2, ...,$
$$\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n.$$
For example,
$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t.$$
- $$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}.$$
- $$\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin(kt).$$
- $$\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt).$$

(Example 1-2 and Exercises 1-30)

In order to use Laplace transform to solve IVPs, we need to evaluate the Laplace transforms of derivatives: Write $\mathcal{L}\{y\} = Y(s)$. Then
$$\mathcal{L}\{y'\} = sY(s) - y(0),$$
and
$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0).$$
Give an IVP of $y(t)$.

(1). Step 1. Take Laplace transform on both sides of the DE. The DE then is transformed to an algebraic equations of $Y(s)$. Here, we need to use the above two formulas and the initial conditions.
(2). Step 2. Solve for $Y(s)$ for the algebraic equation.
(3). Step 3. Take inverse Laplace transform of $Y(s)$ to get $y(t)$ as the solution to the IVP. We usually need to use partial fractions here. See the following example.

(Example 4-5 and Exercises 35-42, 45-46)

Remark.

- Laplace transform can be used to solve IVPs only, i.e. differential equations with initial conditions. Whereas, the methods of Undetermined Coefficients and Variation of Parameters apply to differential equations with or without initial conditions.
- The initial conditions are applied in Step 1. Taking the Laplace transform. So we do NOT apply these conditions in the end.

Problem 13 (Exercise 41 in §7.2)**.**
Consider
$$y'' + y = \sqrt{2}\sin(\sqrt{2}t), \quad y(0) = 10, \; y'(0) = 0.$$
(a). Take Laplace transform of the above, and solve for $Y(s)$.
(b). Use $Y(s)$ to solve for $y(t)$.

Answer.

(a). Take Laplace transform on both sides of the DEs:

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \sqrt{2}\mathcal{L}\{\sin(\sqrt{2}t)\}.$$

Denote $Y(s) = \mathcal{L}\{y\}$. Then

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 10s,$$

by the initial conditions that $y(0) = 10$ and $y'(0) = 0$. So the DE becomes

$$s^2Y(s) - 10s + Y(s) = \sqrt{2}\frac{\sqrt{2}}{s^2 + (\sqrt{2})^2} = \frac{2}{s^2+2}.$$

Solve for $Y(s)$ in the above algebraic equation

$$Y(s) = \frac{2}{(s^2+1)(s^2+2)} + \frac{10s}{s^2+1}.$$

(b). Take inverse Laplace transform of $Y(s)$ to get $y(t)$ as the solution to the IVP. That is,

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{(s^2+1)(s^2+2)} + \frac{10s}{s^2+1}\right\}.$$

Use partial fractions:

$$\frac{2}{(s^2+1)(s^2+2)} = \frac{A}{s^2+1} + \frac{B}{s^2+2} = \frac{A(s^2+2)+B(s^2+1)}{(s^2+1)(s^2+2)} = \frac{(A+B)s^2+2A+B}{(s^2+1)(s^2+2)}.$$

Hence,

$$A + B = 0 \quad \text{and} \quad 2A + B = 2.$$

Solve for A and B: $A = 2$ and $B = -2$. So

$$\frac{2}{(s^2+1)(s^2+2)} = \frac{2}{s^2+1} + \frac{-2}{s^2+2},$$

and

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{2}{(s^2+1)(s^2+2)} + \frac{10s}{s^2+1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{2}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{-2}{s^2+2}\right\} + \mathcal{L}^{-1}\left\{\frac{10s}{s^2+1}\right\} \\ &= 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \sqrt{2}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right\} + 10\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} \\ &= 2\sin(t) - \sqrt{2}\sin(\sqrt{2}t) + 10\cos(t). \end{aligned}$$

7.3. Operational Properties I: Unit Step Functions. Define the unit step function

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a; \\ 1, & t \ge a. \end{cases}$$

Then

$$\begin{aligned}\mathcal{L}\{\mathcal{U}(t-a)\} &= \int_0^\infty e^{-st}\mathcal{U}(t-a)\,dt \\ &= \int_a^\infty e^{-st}\,dt \\ &= \frac{e^{-st}}{-s}\Big|_{t=a}^{t=\infty} \\ &= \frac{e^{-as}}{s}.\end{aligned}$$

(Examples 6 and Exercises 61-62)

Problem 14.
Consider

$$f(t) = \begin{cases} 0, & 0 \le t < 2; \\ 3, & 2 \le t < 5; \\ 0, & t \ge 5. \end{cases}$$

(a). Find $F(s) = \mathcal{L}\{f\}$ using the definition of Laplace transform. (Evaluate an integral.)
(b). Find $F(s) = \mathcal{L}\{f\}$ using the unit step functions.

***Answer*.**
(a). Evaluate an integral:

$$\begin{aligned}\mathcal{L}\{f\} &= \int_0^\infty e^{-st}f(t)\,dt \\ &= \int_2^5 3e^{-st}\,dt \\ &= \frac{3e^{-st}}{-s}\Big|_{t=2}^{t=5} \\ &= \frac{3e^{-2s}}{s} - \frac{3e^{-5s}}{s}.\end{aligned}$$

(b). Represent $f(t)$ by unit step functions:

$$f(t) = 3\mathcal{U}(t-2) - 3\mathcal{U}(t-5).$$

Then

$$\mathcal{L}\{f\} = 3\mathcal{L}\{\mathcal{U}(t-2)\} - 3\mathcal{L}\{\mathcal{U}(t-5)\} = \frac{3e^{-2s}}{s} - \frac{3e^{-5s}}{s}.$$

7.4. Operational Properties II: Laplace Transforms of Integrals. Let $Y(s) = \mathcal{L}\{y\}$. Then

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{Y(s)}{s}.$$

(Exercises 45, 49-52)

Problem 15 (Exercise 49 in §7.4)**.**
Consider

$$y'(t) = 1 - \sin t - \int_0^t y(\tau)\,d\tau, \quad y(0) = 0.$$

(a). Take Laplace transform of the above, and solve for $Y(s)$.
(b). Use $Y(s)$ to solve for $y(t)$.

You may need

$$\mathcal{L}\{t\sin(kt)\} = \frac{2ks}{(s^2+k^2)^2}.$$

Answer.
(a). Take Laplace transform on both sides of the DEs:

$$\mathcal{L}\{y'\} = \mathcal{L}\{1\} - \mathcal{L}\{\sin t\} - \mathcal{L}\left\{\int_0^t y(\tau)\,d\tau\right\}.$$

Denote $Y(s) = \mathcal{L}\{y\}$. Then

$$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s),$$

by the initial conditions that $y(0) = 0$. So the DE becomes

$$sY(s) = \frac{1}{s} - \frac{1}{s^2+1} - \frac{Y(s)}{s}.$$

Solve for $Y(s)$ in the above algebraic equation

$$Y(s) = \frac{1}{s^2+1} - \frac{s}{(s^2+1)^2}.$$

(b). Take inverse Laplace transform of $Y(s)$ to get $y(t)$ as the solution to the IVP. That is,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} \\ &= \sin t - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2s}{(s^2+1)^2}\right\} \\ &= \sin t - \frac{1}{2}t\sin t. \end{aligned}$$

Here, we use the formula provided in the problem ($k = 1$):

$$\mathcal{L}\{t\sin t\} = \frac{2s}{(s^2+1)^2}.$$

7.5. The Dirac Delta Function. The Laplace transform of the Dirac delta function

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}.$$

(Example 1 and Exercises 1-12.)

Problem 16 (Exercise 3 in §7.5)**.**
Consider

$$y'' + y = \delta(t-2\pi), \quad y(0) = 0,\ y'(0) = 1.$$

(a). Take Laplace transform of the above, and solve for $Y(s)$.
(b). Use $Y(s)$ to solve for $y(t)$. You may need

$$\mathcal{L}\{f(t-a)\cdot\mathcal{U}(t-a)\} = \mathcal{L}\{f\}e^{-as}.$$

Answer.
(a). Take Laplace transform on both sides of the DEs:

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\delta(t-2\pi)\}.$$

Denote $Y(s) = \mathcal{L}\{y\}$. Then

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 1,$$

by the initial conditions that $y(0) = 0$ and $y'(0) = 1$. So the DE becomes

$$s^2 Y(s) - 1 + Y(s) = e^{-2\pi s}.$$

Solve for $Y(s)$ in the above algebraic equation

$$Y(s) = \frac{1}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1}.$$

(b). Take inverse Laplace transform of $Y(s)$ to get $y(t)$ as the solution to the IVP. That is,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+1}\right\} \\ &= \sin t + \sin(t - 2\pi) \cdot \mathcal{U}(t - 2\pi) \\ &= \sin t + \sin(t) \cdot \mathcal{U}(t - 2\pi) \end{aligned}$$

Here, we use the formula provided in the problem ($f(t) = \sin t$ and $a = 2\pi$):

$$\mathcal{L}\{\sin(t - 2\pi)\mathcal{U}(t - 2\pi)\} = \frac{e^{-2\pi s}}{s^2+1}.$$

Chapter 8. Systems of Linear First Order Differential Equations

Important topics in this chapter:

(i). Use Wronskian to check two vector functions are linearly independent.
(ii). Use matrix form to solve homogeneous systems of linear DEs.

8.1. Preliminary Theory – Linear Systems. Give two vector functions

$$\overrightarrow{X}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{X}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

Their Wronskian

$$W(\overrightarrow{X}_1, \overrightarrow{X}_2) = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}.$$

If the Wronskian $W(\overrightarrow{X}_1, \overrightarrow{X}_2) \neq 0$, then we say the two vector functions $\overrightarrow{X}_1$ and $\overrightarrow{X}_2$ are linearly independent. (Exercises 17-18)

8.2. Homogeneous Linear Systems. Give a homogeneous linear system of first order DEs with constant coefficients

$$\begin{cases} \frac{dx}{dt} = ax + by, \\ \frac{dy}{dt} = cx + dy. \end{cases}$$

Here, a, b, c, d are numbers.

(1). Step 1. Write the system in matrix form.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Denote

$$\overrightarrow{X} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then

$$\overrightarrow{X}' = A\overrightarrow{X}, \quad \text{in which the matrix } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(2). Find the eigenvalues and eigenvectors. The eigenvalues are the solutions to the characteristic equation of the matrix A:

$$\det(A - \lambda I) = 0.$$

- Case 1. A has two distinct real eigenvalues λ_1 and λ_2 with eigenvectors $\overrightarrow{K}_1$ and $\overrightarrow{K}_2$, respectively. The eigenvectors are the solutions to

  $$(A - \lambda_1 I)\overrightarrow{K}_1 = 0 \quad \text{and} \quad (A - \lambda_2 I)\overrightarrow{K}_2 = 0.$$

  Then the general solution

  $$\overrightarrow{X} = c_1 e^{\lambda_1 t}\overrightarrow{K}_1 + c_2 e^{\lambda_2 t}\overrightarrow{K}_2.$$

  (Example 1 and Exercises 1-6, 13)
- Case 2. A has two equal real eigenvalues λ (i.e. repeated eigenvalues) with eigenvector $\overrightarrow{K}$ and generalized eigenvector $\overrightarrow{P}$.

  $$(A - \lambda I)\overrightarrow{K} = 0 \quad \text{and} \quad (A - \lambda I)\overrightarrow{P} = \overrightarrow{K}.$$

  Then the general solution

  $$\overrightarrow{X} = c_1 e^{\lambda t}\overrightarrow{K} + c_2\big[te^{\lambda t}\overrightarrow{K} + e^{\lambda t}\overrightarrow{P}\big].$$

  (Example 4 and Exercises 21-24, 31)
- Case 3. A has two complex conjugate eigenvalues $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$. The eigenvalue $\lambda_1 = \alpha + i\beta$ has real column vectors $\overrightarrow{B}_1$ and $\overrightarrow{B}_2$ from a complex eigenvector. That is, find the eigenvector $\overrightarrow{K}$ that

  $$(A - \lambda_1 I)\overrightarrow{K} = 0.$$

  We have that

  $$\overrightarrow{K} = \overrightarrow{B}_1 + i\overrightarrow{B}_2.$$

  Notice that we need to use $\lambda_1 = \alpha + i\beta$ (instead of $\lambda_2 = \alpha - i\beta$) to find the eigenvectors to write the general solution

  $$\overrightarrow{X} = c_1 e^{\alpha t}\left[\cos(\beta t)\overrightarrow{B}_1 - \sin(\beta t)\overrightarrow{B}_2\right] + c_2 e^{\alpha t}\left[\cos(\beta t)\overrightarrow{B}_2 + \sin(\beta t)\overrightarrow{B}_1\right].$$

  (Example 6 and Exercises 35-40, 48)

(3). Step 3. We only do this step if the problem is an initial value problem (IVP). That is, if there is initial condition $\overrightarrow{X}(t_0) = \overrightarrow{X}_0$, then plug them into the general solution to find the constants c_1 and c_2.

Problem 17 (Exercise 1 in §8.2)**.**
Consider the system of ODE's

$$\begin{cases} \frac{dx}{dt} = x + 2y, \\ \frac{dy}{dt} = 4x + 3y. \end{cases}$$

(a). Write as a matrix equation $\overrightarrow{X}' = A\overrightarrow{X}$*.*
(b). Find $\det(A - \lambda I)$*.*
(c). Find the eigenvalues and eigenvectors of A*.*
(d). Use the above to find the general solution to the system.
(e). Find the solution with $\overrightarrow{X}(0) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$*.*

Answer.

(a). The matrix equation is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(b).

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5.$$

(c). The eigenvalues are the solutions to

$$\det(A - \lambda I) = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) = 0.$$

So the two eigenvalues are

$$\lambda_1 = 5 \quad \text{and} \quad \lambda_2 = -1.$$

The eigenvector $\overrightarrow{K}_1$ corresponding to the eigenvalue $\lambda_1 = 5$ solves

$$(A - \lambda_1 I)\overrightarrow{K} = 0,$$

that is,

$$\begin{pmatrix} 1-\lambda_1 & 2 \\ 4 & 3-\lambda_1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = 0.$$

Hence,

$$\begin{cases} -4r_1 + 2r_2 = 0, \\ 4r_1 - 2r_2 = 0. \end{cases}$$

Set $r_1 = 1$. Then $r_2 = 2$ and the eigenvector

$$\overrightarrow{K}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The eigenvector $\overrightarrow{K}_2$ corresponding to the eigenvalue $\lambda_2 = -1$ solves

$$(A - \lambda_2 I)\overrightarrow{K} = 0,$$

that is,

$$\begin{pmatrix} 1-\lambda_2 & 2 \\ 4 & 3-\lambda_2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = 0.$$

Hence,

$$\begin{cases} 2p_1 + 2p_2 = 0, \\ 4p_1 + 4p_2 = 0. \end{cases}$$

Set $p_2 = 1$. Then $p_1 = -1$ and the eigenvector

$$\overrightarrow{K}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

(d). The general solution to the system is

$$\overrightarrow{X} = c_1 e^{5t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

(e). Using the initial condition at $t = 0$,

$$c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

That is,

$$\begin{cases} c_1 - c_2 = 0, \\ 2c_1 + c_2 = 3. \end{cases}$$

Solve for c_1 and c_2:

$$c_1 = 1 \quad \text{and} \quad c_2 = 1.$$

Hence, the solution is

$$\overrightarrow{X} = e^{5t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{5t} - e^{-t} \\ 2e^{5t} + e^{-t} \end{pmatrix}.$$

Problem 18 (Exercise 31 in §8.2)**.**
Consider the system of ODE's

$$\begin{cases} \frac{dx}{dt} = 2x + 4y, \\ \frac{dy}{dt} = -x + 6y. \end{cases}$$

(a). Write as a matrix equation $\overrightarrow{X}' = A\overrightarrow{X}$.
(b). Find $\det(A - \lambda I)$.
(c). Find the eigenvalues and eigenvectors of A.
(d). Use the above to find the general solution to the system.
(e). Find the solution with $\overrightarrow{X}(0) = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$.

Answer.
(a). The matrix equation is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

(b).

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 4 \\ -1 & 6-\lambda \end{vmatrix} = (2-\lambda)(6-\lambda) + 4 = \lambda^2 - 8\lambda + 16.$$

(c). The eigenvalues are the solutions to

$$\det(A - \lambda I) = \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2 = 0.$$

So the repeated eigenvalues are

$$\lambda = 4.$$

The eigenvector $\overrightarrow{K}$ corresponding to the eigenvalue $\lambda = 4$ solves

$$(A - \lambda I)\overrightarrow{K} = 0,$$

that is,

$$\begin{pmatrix} 2-\lambda & 4 \\ -1 & 6-\lambda \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = 0.$$

Hence,

$$\begin{cases} -2r_1 + 4r_2 = 0, \\ -r_1 + 2r_2 = 0. \end{cases}$$

Set $r_2 = 1$. Then $r_1 = 2$ and the eigenvector

$$\overrightarrow{K} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The generalized eigenvector $\overrightarrow{P}$ corresponding to the eigenvalue $\lambda = 4$ solves

$$(A - \lambda I)\overrightarrow{P} = \overrightarrow{K},$$

that is,

$$\begin{pmatrix} 2-\lambda & 4 \\ -1 & 6-\lambda \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \overrightarrow{K}.$$

Hence,

$$\begin{cases} -2p_1 + 4p_2 = 2, \\ -p_1 + 2p_2 = 1. \end{cases}$$

Set $p_2 = 1$. Then $p_1 = 1$ and the generalized eigenvector

$$\overrightarrow{P} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(d). The general solution to the system is

$$\overrightarrow{X} = c_1 e^{4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \left[t e^{4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right].$$

(e). Using the initial condition at $t = 0$,

$$c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}.$$

That is,

$$\begin{cases} 2c_1 + c_2 = -1, \\ c_1 + c_2 = 6. \end{cases}$$

Solve for c_1 and c_2:

$$c_1 = -7 \quad \text{and} \quad c_2 = 13.$$

Hence, the solution is

$$\overrightarrow{X} = -7e^{4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 13 \left[t e^{4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] = \begin{pmatrix} 26te^{4t} - e^{4t} \\ 13te^{4t} + 6e^{4t} \end{pmatrix}.$$

Problem 19 (Exercise 48 in §8.2)**.**
Consider the system of ODE's

$$\begin{cases} \frac{dx}{dt} = 6x - y, \\ \frac{dy}{dt} = 5x + 4y. \end{cases}$$

(a). Write as a matrix equation $\overrightarrow{X}' = A\overrightarrow{X}$.
(b). Find $\det(A - \lambda I)$.
(c). Find the eigenvalues and eigenvectors of A.
(d). Use the above to find the general solution to the system.
(e). Find the solution with $\overrightarrow{X}(0) = \begin{pmatrix} -2 \\ 8 \end{pmatrix}$.

Answer.
(a). The matrix equation is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(b).

$$\det(A-\lambda I) = \begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} = (6-\lambda)(4-\lambda)+5 = \lambda^2 - 10\lambda + 29.$$

(c). The eigenvalues are the solutions to

$$\det(A-\lambda I) = \lambda^2 - 10\lambda + 29 = (\lambda-5)^2 + 4 = 0.$$

So the two complex eigenvalues are

$$\lambda_1 = 5+2i \quad \text{and} \quad \lambda_2 = 5-2i.$$

The eigenvector $\overrightarrow{K}$ corresponding to the eigenvalue $\lambda_1 = 5+2i$ solves

$$(A-\lambda I)\overrightarrow{K} = 0,$$

that is,

$$\begin{pmatrix} 6-\lambda_1 & -1 \\ 5 & 4-\lambda_1 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 1-2i & -1 \\ 5 & -1-2i \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = 0.$$

Hence,

$$\begin{cases} (1-2i)r_1 - r_2 = 0, \\ 5r_1 - (1+2i)r_2 = 0. \end{cases}$$

Set $r_1 = 1$. Then $r_2 = 1-2i$ and the eigenvector

$$\overrightarrow{K} = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

The two real column vectors corresponding to $\lambda_1 = 5+2i$ are

$$\overrightarrow{B}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{B}_2 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

(d). The general solution to the system is

$$\begin{aligned} \overrightarrow{X} &= c_1 e^{5t}\left[\cos(2t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin(2t)\begin{pmatrix} 0 \\ -2 \end{pmatrix}\right] + c_2 e^{5t}\left[\cos(2t)\begin{pmatrix} 0 \\ -2 \end{pmatrix} + \sin(2t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] \\ &= c_1 e^{5t}\begin{pmatrix} \cos(2t) \\ \cos(2t) + 2\sin(2t) \end{pmatrix} + c_2 e^{5t}\begin{pmatrix} \sin(2t) \\ -2\cos(2t) + \sin(2t) \end{pmatrix}. \end{aligned}$$

(e). Using the initial condition at $t = 0$,

$$c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 8 \end{pmatrix}.$$

That is,

$$\begin{cases} c_1 = -2, \\ c_1 - 2c_2 = 8. \end{cases}$$

Solve for c_1 and c_2:

$$c_1 = -2 \quad \text{and} \quad c_2 = -5.$$

Hence, the solution is

$$\begin{aligned} \overrightarrow{X} &= -2e^{5t}\begin{pmatrix} \cos(2t) \\ \cos(2t) + 2\sin(2t) \end{pmatrix} - 5e^{5t}\begin{pmatrix} \sin(2t) \\ -2\cos(2t) + \sin(2t) \end{pmatrix} \\ &= e^{5t}\begin{pmatrix} -2\cos(2t) - 5\sin(2t) \\ 8\cos(2t) - 9\sin(2t) \end{pmatrix}. \end{aligned}$$

Appendix: Table of Formulas

Laplace Transforms. Write $F = \mathcal{L}\{f\}$ as the Laplace transform of f in the following formulas.

- $$\mathcal{L}\{1\} = \frac{1}{s}.$$
- $$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}.$$
- $$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}.$$
- $$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}.$$
- $$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}.$$
- $$\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}.$$
- $$\mathcal{L}\{e^{at}\sin(\omega t)\} = \frac{\omega}{(s-a)^2+\omega^2}.$$
- $$\mathcal{L}\{e^{at}\cos(\omega t)\} = \frac{s-a}{(s-a)^2+\omega^2}.$$
- $$\mathcal{L}\{e^{at}f(t)\} = F(s-a).$$
- $$\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{1}{s}e^{-as}.$$
- $$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = F(s)e^{-as}.$$
- $$\mathcal{L}\{f(t)\mathcal{U}(t-a)\} = \mathcal{L}\{f(t+a)\}e^{-as}.$$
- $$\mathcal{L}\left\{\frac{d^n}{dt^n}f(t)\right\} = s^nF(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Linear Models.

- Newton's Law of cooling/warming. (Example 4 in §3.1)
$$\frac{dT}{dt} = k(T - T_m).$$
- Series circuits. (Example 7 in §3.1)
$$L\frac{d\mathbf{i}}{dt} + R\mathbf{i} = E(t).$$
- Air resistance. (Exercise 35 in §3.1)
$$m\frac{dv}{dt} = mg - kv.$$
- Spring/mass systems.
$$m\frac{dx^2}{dt^2} + \beta\frac{dx}{dt} + kx = F(t).$$