STUDY GUIDE OF *CALCULUS III*

ABSTRACT. This Study Guide includes the important topics and problems that are featured in the Tests and the Final of *Calculus III*. You are entitled to a reward of 2 points toward a Test if you are the first person to report a mathematical mistake.

Remark. There are no choice questions in the Tests and Final. So when you review the choice questions in the online homework, understand the correct answer and make sure you can answer the question without all the choices being given.

CONTENTS

CHAPTER 13. FUNCTIONS OF SEVERAL VARIABLES

(i). Compute the derivatives, including applications of chain rules and implicit differentiation.
(ii). Graph the six quadric surfaces in $\mathbb{R}^3$; draw the level curves and contour map on $\mathbb{R}^2$; derive the equation of the tangent planes to these surfaces using partial derivatives.
(iii). Compute the limit of a function at a point if the point is in the domain of the function (and therefore the limit exists); prove the limit does not exist by choosing two paths.

(iv). Compute the directional derivative $D_{\vec{u}}f$, where $\vec{u}$ is a unit directional vector and $D_{\vec{u}}f$ is a rate of change; compute the gradient vector ∇f and use it to find the maximal and minimal rates of change and the corresponding directions $\vec{u}_{\max}$ and $\vec{u}_{\min}$.
(v). Use first-order partial derivative to find the linear approximation of a function.
(vi). Use first-order partial derivative to find the critical points and use second-order partial derivatives to classify them as local maximum, local minimum, saddle points, or inconclusive.

§13.1. Planes and surfaces.

- Equations of planes in $\mathbb{R}^3$.
- Graphs of quadric surfaces through intercepts and traces: ellipsoid, elliptic paraboloid, hyperboloid of one sheet, hyperboloid of two sheets, elliptic cone, and hyperbolic paraboloid.
- Important online questions: 2, 3, 9, 10, 12, 14, 17, 19.

§13.2. Graphs and level curves.

- Domain and range of a function $f(x, y)$ with two independent variables x and y.
- Level curves and contour map of a function.

  *The contour maps consisting of the level curves are on the xy-plane, **not** in the xyz-space!*
- Important online questions: 1, 4, 5, 7.

§13.3. Limit of a function of two variables.

- If (a, b) is in the domain of the function f and f is continuous at (a, b), then
$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$
  The polynomial, rational, exponential, radical, trigonometric functions are all continuous in their natural domains. This means that, if the point is in the domain of such a function, then the limit of the function approaching the point is simply the function value at this point.
- To prove
$$\lim_{(x,y)\to(a,b)} f(x, y) \text{ does not exist,}$$
  one needs to construct two paths of $(x, y) \to (a, b)$ such that there are two distinct limits along the two paths.
- Important online questions: 1, 4, 5, 7, 8.

§13.4. Partial derivatives.

- Compute the partial derivatives.
- Clairaut Theorem: $f_{xy} = f_{yx}$.
- Important online questions: 1, 5, 6, 10, 15.

§13.5. Chain rule.

- Chain rule with one independent variable: $z = f(x, y)$, $x = x(t)$, and $y = y(t)$. Then
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$
- Chain rule with two independent variables: $z = f(x, y)$, $x = x(s, t)$, and $y = y(s, t)$. Then
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$
- Chain rule can be generalized to more independent variables.

- Implicit differentiation: If $F(x,y,z)=0$, then
$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}.$$
*Remember the **minus sign** in the above formulas!*
- Important online questions: 3, 4, 7, 8, 13, 14.

§13.6. Directional derivative and the gradient.

- Let $f=f(x,y)$ be differentiable at (a,b) and $\vec{u}=\langle u_1,u_2\rangle$ be a unit vector in $\mathbb{R}^2$. Then the directional derivative of f at (a,b) in the direction of $\vec{u}$ is
$$D_{\vec{u}}f(a,b)=u_1f_x(a,b)+u_2f_y(a,b)=\langle f_x(a,b),f_y(a,b)\rangle\cdot\langle u_1,u_2\rangle=\nabla f\cdot\vec{u}.$$
*The directional derivative is a **scalar**!*
- The gradient of f at (a,b) is
$$\nabla f(a,b)=\langle f_x(a,b),f_y(a,b)\rangle=f_x(a,b)\vec{i}+f_y(a,b)\vec{j}.$$
*The gradient is a **vector** and it is always orthogonal to the level curves of the function.*
- Let $f=f(x,y,z)$ be differentiable at (a,b,c) and $\vec{u}=\langle u_1,u_2,u_3\rangle$ be a unit vector in $\mathbb{R}^3$. Then the directional derivative of f at (a,b,c) in the direction of $\vec{u}$ is
$$D_{\vec{u}}f(a,b,c)=\langle f_x(a,b,c),f_y(a,b,c),f_z(a,b,c)\rangle\cdot\langle u_1,u_2,u_3\rangle=\nabla f\cdot\vec{u},$$
in which
$$\nabla f(a,b,c)=\langle f_x(a,b,c),f_y(a,b,c),f_z(a,b,c)\rangle=f_x(a,b,c)\vec{i}+f_y(a,b,c)\vec{j}+f_z(a,b,c)\vec{k}.$$
- At (a,b), the maximal directional derivative is achieved when the directional vector
$$\vec{u}_{\max}=\frac{\nabla f(a,b)}{|\nabla f(a,b)|},$$
and
$$D_{\vec{u}_{\max}}f(a,b)=|\nabla f(a,b)|,$$
while the minimal directional derivative is achieved when the directional vector
$$\vec{u}_{\min}=-\frac{\nabla f(a,b)}{|\nabla f(a,b)|},$$
and
$$D_{\vec{u}_{\min}}f(a,b)=-|\nabla f(a,b)|.$$
*Keep in mind that $\vec{u}_{\max}$ and $\vec{u}_{\min}$ are two (unit) directional **vectors** while $D_{\vec{u}_{\max}}f(a,b)$ and $D_{\vec{u}_{\min}}f(a,b)$ are two **scalars**!*
- If $\vec{u}$ is parallel to ∇f, then $D_{\vec{u}}f=0$ and there is no change of the function in the direction of $\vec{u}$.
- Important online questions: 3, 5, 9, 10, 12, 15, 18, 21, 22.

§13.7. Tangent planes and linear approximation.

- The tangent plane to $F(x,y,z)=0$ at (a,b,c) has the equation
$$F_x(a,b,c)(x-a)+F_y(a,b,c)(y-b)+F_z(a,b,c)(z-c)=0.$$
- The tangent plane to $z=f(x,y)$ at $(a,b,f(a,b))$ has the equation
$$z=f_x(a,b)(x-a)+f_y(a,b)(y-b)+f(a,b).$$

- Around the base point (a, b), the function $f(x, y)$ can be approximated by the linear approximation
$$f(x, y) \approx L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$
Keep in mind that the partial differentiation f_x and f_y here are evaluated ***at the base point*** (a, b) *and remember the term $f(a, b)$.*
- The total differential
$$dz = f_x(a, b)dx + f_y(a, b)dy.$$
- Important online questions: 1, 2, 3, 4, 5, 8, 9, 13.

§13.8. Optimization problems.

- The critical points (a, b) of $f(x, y)$ satisfies
$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0.$$
- Second derivative test: Let $D(x, y) = f_{xx}f_{yy} - f_{xy}^2$. Suppose that (a, b) is a critical point of f.
  - If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum value at (a, b);
  - If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum value at (a, b);
  - If $D(a, b) < 0$, then f has a saddle point at (a, b);
  - If $D(a, b) = 0$, then the test is inconclusive.
- Find the absolute maximal and minimal values of a function f on a bounded domain R:
  (1) Determine the values of f at all critical points in R;
  (2) Find the maximum and minimal values of f on the boundary of R;
  (3) The absolute maximal value is the greatest value in the above two steps, and the absolute minimal value is the least value in the above two steps.
- Important online questions: 1, 3, 4, 9, 12, 18, 20.

Chapter 14. Multiple integration

(i). Evaluate double integrals
$$\iint_R f(x, y)\, dA,$$
where R is rectangular, y-simple, x-simple, polar rectangular, r-simple in polar coordinates. In particular, use polar if the function f contains $x^2 + y^2$ or the the domain R is a piece of a disc.

(ii). The area of a region R in $\mathbb{R}^2$ is $\iint_R 1\, dA$.

(iii). Evaluate triple integrals
$$\iint_D f(x, y, z)\, dV,$$
where R is cubic, z-simple, y-simple, x-simple, in cylindrical coordinates (r, θ, z), in spherical coordinates (ρ, φ, θ). In particular, use cylindrical if the function f contains $x^2 + y^2$; use spherical is the function f contains $x^2 + y^2 + z^2$ or the domain D is a piece of a ball.

(iv). The volume of a solid D in $\mathbb{R}^3$ is $\iint_D 1\, dV$.

(v). The extra factor in polar and cylindrical is r; while the extra factor in spherical is $\rho^2 \sin\varphi$.

(vi). Use change of variables to evaluate double integral and remember the absolute value of the Jacobian.

§14.1. Double integrals over rectangular regions.

- If R is rectangular, i.e. $R = [a, b] \times [c, d] = \{(x, y) \mid a \le x \le b, c \le y \le d\}$, then
$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dxdy = \int_a^b \int_c^d f(x, y)\, dydx.$$
If the region is rectangular, then one can change the order of integration freely as long as one keeps the bounds of x and y correspondingly.
- Important online questions: 4, 7, 8, 14.

§14.2. Double integrals over general regions.

- If R is y-simple, i.e. $R = \{(x, y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$, then
$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dydx.$$
If the region is y-simple, then do y-integration first!
- If R is x-simple, i.e. $R = \{(x, y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$, then
$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dxdy.$$
If the region is x-simple, then do x-integration first!
- One can change the order of integration between y-simple and x-simple. Always sketch the integrating region and find all the required information.
- The area of a region R in $\mathbb{R}^2$ is
$$\text{Area}\,(R) = \iint_R 1\, dA.$$
- The volume of a solid in $\mathbb{R}^3$ above a region R in $\mathbb{R}^2$ and is bounded above by $z = h_1(x, y)$ and bounded below by $z = h_2(x, y)$ is
$$\iint_R \left[h_1(x, y) - h_2(x, y)\right]\, dA.$$
- Important online questions: 9, 13, 15, 17, 18, 19.

§14.3. Double integrals in polar coordinates.

- If R is polar rectangular, i.e. $R = \{(r, \theta) \mid \alpha \le \theta \le \beta, 0 \le a \le r \le b\}$, then
$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\, drd\theta.$$
Remember the ***extra factor*** *r; use polar coordinates if the integrating function f contains $x^2 + y^2$ or the integrating region R is a piece of a disc.*
- If R is r-simple in polar coordinates, i.e. $R = \{(r, \theta) \mid \alpha \le \theta \le \beta, 0 \le g_1(\theta) \le r \le g_2(\theta)\}$, then
$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta) r\, drd\theta.$$
- Important online questions: 1, 2, 3, 5, 8, 12, 14.

§14.4. Triple integrals.

- If D is cubic, i.e. $D = [a,b] \times [c,d] \times [p,q] = \{(x,y,z) \,|\, a \le x \le b, c \le y \le d, p \le z \le q\}$, then
$$\iiint_D f(x,y,z)\,dV = \int_p^q \int_c^d \int_a^b f(x,y,z)\,dxdydz = \cdots .$$
If the region is cubic, then one can change within the ***six*** *orders of integration freely as long as one keeps the bounds of x, y, and z correspondingly.*
- If D is z-simple, i.e. $D = \{(x,y) \,|\, (x,y) \in R, h_1(x,y) \le z \le h_2(x,y)\}$, then
$$\iiint_D f(x,y,z)\,dV = \iint_R \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dzdA.$$
If the region is z-simple, then do z-integration first!
- There are also y-simple and x-simple regions.
- The volume of a solid D is
$$\iiint_D 1dV.$$
- Important online questions: 4, 6, 9, 10.

§14.5. Triple integrals in cylindrical and spherical coordinates.

- If $D = \{(r,\theta,z) \,|\, \alpha \le \theta \le \beta, 0 \le a \le r \le b, h_1(r,\theta) \le z \le h_2(r,\theta)\}$, then
$$\iiint_D f(x,y,z)\,dV = \int_\alpha^\beta \int_a^b \int_{h_1(r,\theta)}^{h_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\,dzr\,drd\theta.$$
Remember the ***extra factor*** *r; use cylindrical coordinates if the integrating function f contains x^2+y^2 and cylindrical is simply polar plus the z-variable.*
- In spherical coordinates, a point P in $\mathbb{R}^3$ can be represented as $P = (\rho,\varphi,\theta)$.
  - ρ is the distance from the origin to P, hence $\rho \ge 0$;
  - φ is the angle between the positive z-axis and the ray $\overrightarrow{OP}$, hence, $0 \le \varphi \le \pi$;
  - θ is the angle that measures the rotation with respect to the z-axis, hence, $0 \le \theta \le 2\pi$.
- If D is a spherical rectangle, i.e. $D = \{(\rho,\varphi,\theta) \,|\alpha \le \theta \le \beta, p \le \varphi \le q, a \le \rho \le b\}$, then
$$\iiint_D f(x,y,z)\,dV = \int_\alpha^\beta \int_p^q \int_a^b f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi\,d\rho d\varphi d\theta.$$
Remember the ***extra factor*** *$\rho^2\sin\varphi$; use spherical coordinates if the integrating function f contains $x^2+y^2+z^2$ or the integration region is a piece of a ball.*
- If D is ρ-simple in spherical coordinates, i.e. $D = \{(\rho,\varphi,\theta) \,|\alpha \le \theta \le \beta, p \le \varphi \le q, h_1(\varphi,\theta) \le \rho \le h_2(\varphi,\theta)\}$, then
$$\iiint_D f(x,y,z)\,dV = \int_\alpha^\beta \int_p^q \int_{h_1(\varphi,\theta)}^{h_2(\varphi,\theta)} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi\,d\rho d\varphi d\theta.$$
- Important online questions: 3, 4, 5, 10, 12.

§14.7. Change of variables.

- If $x = x(u,v)$ and $y = y(u,v)$, then
$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA,$$

in which

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

is the Jacobian determinant. *Keep in mind the* ***extra factor is the absolute value of the Jacobian determinant****, and you need to find the correct integrating domain in* (u, v).

- If you have u and v as functions of x and y, then you need to solve for x and y in order to compute the Jacobian!
- Important online questions: 3, 6, 7, 12,

Chapter 15. Vector calculus

Parametric equation of a curve is a vector function $\vec{r}(t)$.

(i). $C : \vec{r}(t) = \langle x(t), y(t)\rangle$, where $a \le t \le b$, defines a curve in $\mathbb{R}^2$ with an orientation.

(ii). $C : \vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, where $a \le t \le b$, defines a curve in $\mathbb{R}^3$ with an orientation.

The arclength of C is

$$\int_a^b |\overrightarrow{r}'(t)|\, dt.$$

Important curves: straight line, circle, ellipse, helix, etc.

Two types of vector field: $\vec{F} = \langle f(x, y), g(x, y)\rangle$ in $\mathbb{R}^2$ and $\vec{F} = \langle f(x, y, z), g(x, y, z), h(x, y, z)\rangle$ in $\mathbb{R}^3$. We can discuss two topics for both of them:

(iii). Is $\vec{F}$ conservative? There are several equivalent criteria. If it is, then find its potential function φ such that $\nabla\varphi = \vec{F}$.

(iv). Evaluate the line integrals of $\vec{F}$ in the circulation form along a curve C:

$$\int_C \vec{F}\cdot\vec{T}\, ds = \int_C \vec{F}\cdot\overrightarrow{r}'\, ds = \int_C \vec{F}\cdot\, d\vec{r}.$$

Then we discuss the fundamental theory of calculus in $\mathbb{R}^2$ and in $\mathbb{R}^3$, separately:

(vi). **Fundamental theory of calculus for vector fields in** $\mathbb{R}^2$: A closed curve C enclosing a region R in $\mathbb{R}^2$, Green's Theorem connects **double integral in the interior and line integral on the boundary**.

- Circulation form:

$$\oint_C \vec{F}\cdot\, d\vec{r} = \oint_C f\, dx + g\, dy = \iint_R (g_x - f_y)\, dA.$$

In the special case, when $\vec{F}$ is conservative, $g_x = f_y$ so $\oint_C \vec{F}\cdot\, d\vec{r} = 0$.

- Flux form:

$$\oint_C \vec{F}\cdot\vec{n}\, ds = \oint_C f\, dy - g\, dx = \iint_R (f_x + g_y)\, dA.$$

(vii). Parametric equation of a surface S in $\mathbb{R}^3$:

$$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle, \quad \text{where } (u, v) \in R.$$

The area of S is

$$\iint_S 1\, dS = \iint_R |\overrightarrow{r}'_u \times \overrightarrow{r}'_v|\, dA.$$

Important surfaces: plane, cylinder, cone, sphere, paraboloid, etc.

(viii). **Fundamental theory of calculus for vector fields in $\mathbb{R}^3$. I**: A closed curve C as the boundary of a surface S in $\mathbb{R}^3$, Stokes' Theorem connects **flux surface integral of the curl in the interior and circulation line integral of the vector field on the boundary**:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS.$$

(ix). **Fundamental theory of calculus for vector fields in $\mathbb{R}^3$. II**: A closed surface S enclosing a solid D in $\mathbb{R}^3$, Divergence Theorem connects **triple integral of the divergence in the interior and outward flux surface integral of the vector field on the boundary**:

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_D \nabla \cdot \vec{F}\, dV.$$

§15.1. Vector fields.

- A vector field in $\mathbb{R}^2$:
$$\vec{F}(x, y) = \langle f(x, y), g(x, y)\rangle = f(x, y)\vec{i} + g(x, y)\vec{j}.$$
- A vector field in $\mathbb{R}^3$:
$$\vec{F}(x, y, z) = \langle f(x, y, z), g(x, y, z), h(x, y, z)\rangle = f(x, y, z)\vec{i} + g(x, y, z)\vec{j} + h(x, y, z)\vec{k}.$$
- Recall that the gradient vector field of $\varphi(x, y)$ is defined as
$$\nabla\varphi = \langle \varphi_x, \varphi_y\rangle = \varphi_x\vec{i} + \varphi_y\vec{j},$$
and the gradient vector field of $\varphi(x, y, z)$ is defined as
$$\nabla\varphi = \langle \varphi_x, \varphi_y, \varphi_z\rangle = \varphi_x\vec{i} + \varphi_y\vec{j} + \varphi_z\vec{k}.$$
- Important online questions: 4, 5, 6, 7, 10.

§15.2. Line integrals.

- A curve in $\mathbb{R}^2$ is a vector function
$$\vec{r}(t) = \langle x(t), y(t)\rangle = x(t)\vec{i} + y(t)\vec{j} \quad \text{where } a \le t \le b.$$
The above parametric form of the curve defines an ***orientation*** *along the curve as the direction when t increases from a to b.*
- Parametrize straight line, circle, ellipse, etc.
- The line integral of a scalar function $f(x, y)$ along a curve C is
$$\int_C f\, ds = \int_a^b f(x(t), y(t))|\vec{r}\,'(t)|\, dt,$$
in which
$$\vec{r}\,'(t) = \langle x'(t), y'(t)\rangle \quad \text{and} \quad |\vec{r}\,'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}.$$
- The arclength of C is
$$\int_a^b |\vec{r}\,'(t)|\, dt.$$
- The line integral of a vector field $\vec{F} = \langle f(x, y), g(x, y)\rangle$ along a curve C has two forms:

- ∘ Circulation form:

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot\overrightarrow{r}'\,ds = \int_C \vec{F}\cdot\,d\vec{r} = \int_C f\,dx + g\,dy$$
$$= \int_a^b \langle f(t), g(t)\rangle \cdot \langle x'(t), y'(t)\rangle\,dt = \int_a^b f(t)x'(t) + g(t)y'(t)\,dt.$$

- ∘ Flux form:

$$\int_C \vec{F}\cdot\vec{n}\,ds = \int_C f\,dy - g\,dx$$
$$= \int_a^b f(t)y'(t) - g(t)x'(t)\,dt.$$

- Important online questions: 1, 4, 5, 8, 14, 15.

§15.3. Conservative vector fields.

- The following statements are **equivalent**.
  - ∘ A vector field $\vec{F} = \langle f, g\rangle$ is conservative.
  - ∘ $\vec{F}$ has a potential function, i.e. $\vec{F} = \nabla\varphi$ for some scalar function φ.
  - ∘ $\int_C \vec{F}\cdot\,d\vec{r} = \varphi(B) - \varphi(A)$ if C is a curve from initial point A to terminal point B. That is, the line integral from A to B of a conservative vector field is **path independent**.
  - ∘ $\oint_C \vec{F}\cdot\,d\vec{r} = 0$ if C is a closed curve.
  - ∘ $f_y = g_x$, i.e. curl $\vec{F} = 0$ and $\vec{F}$ is irrotational.
- Find the potential function if a vector field is conservative.
- Important online questions: 1, 2, 3, 6, 10.

§15.4. Green's Theorem.

- Green's Theorem has two forms. Let C be a closed curve enclosing a region R in $\mathbb{R}^2$.
  - ∘ Circulation form:

$$\oint_C \vec{F}\cdot\,d\vec{r} = \oint_C f\,dx + g\,dy = \iint_R (g_x - f_y)\,dA.$$

In the special case, when $\vec{F}$ is conservative, $g_x = f_y$ so $\oint_C \vec{F}\cdot\,d\vec{r} = 0$.

  - ∘ Flux form:

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C f\,dy - g\,dx = \iint_R (f_x + g_y)\,dA.$$

The key observation of Green's Theorem is that ***the double integral in the interior equals the line integral on the boundary****.*

- An application of Green's Theorem is to use circulation to compute the area of R:

$$\text{Area}\,(R) = \iint_R 1\,dA = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C (x\,dy - y\,dx).$$

- Important online questions: 10, 11, 13, 14, 15.

§15.5. Divergence and curl.

- Let $\vec{F}$ be a vector field in $\mathbb{R}^3$:

  $\vec{F}(x,y,z) = \langle f(x,y,z), g(x,y,z), h(x,y,z)\rangle = f(x,y,z)\vec{i} + g(x,y,z)\vec{j} + h(x,y,z)\vec{k}.$

  Then the divergence of $\vec{F}$ is

  $$\operatorname{div}\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = \langle \partial_x, \partial_y, \partial z\rangle \cdot \langle f, g, h\rangle = \nabla \cdot \vec{F},$$

  and the curl of $\vec{F}$ is

  $$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial z \\ f & g & h \end{vmatrix} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\vec{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\vec{i} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\vec{i} = \nabla \times \vec{F}.$$

  The divergence is scalar and the curl is vector. If $\operatorname{div}\vec{F} = 0$, then $\vec{F}$ is **incompressible** (i.e. source-free); if $\operatorname{curl}\vec{F} = 0$, then $\vec{F}$ is **irrotational**.
- The following statements are **equivalent**.
  - A vector field $\vec{F} = \langle f, g, h\rangle$ is conservative.
  - $\vec{F}$ has a potential function, i.e. $\vec{F} = \nabla\varphi$ for some scalar function φ.
  - $\int_C \vec{F}\cdot d\vec{r} = \varphi(B) - \varphi(A)$ if C is a curve from initial point A to terminal point B. That is, the line integral from A to B of a conservative vector field is **path independent**.
  - $\oint_C \vec{F}\cdot d\vec{r} = 0$ if C is a closed curve.
  - $f_y = g_x$, $f_z = h_x$, and $h_y = g_z$, i.e. $\operatorname{curl}\vec{F} = 0$ and $\vec{F}$ is irrotational.
- Find the potential function of a vector field if it is conservative.
- Show that the curl of a gradient vector field is zero, i.e. the vector field conservative:

  $$\operatorname{curl}(\nabla\varphi) = \nabla \times (\nabla\varphi) = 0.$$
- Show that the divergence of a curl is zero:

  $$\operatorname{div}(\operatorname{curl}\vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = 0.$$
- Important online questions: 4, 5, 10, 11.

§15.6. Surface integrals.

- A surface in $\mathbb{R}^3$ is a vector function

  $$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}, \quad \text{where } (u,v) \in R.$$
- Parametrize plane, cylinder, cone, sphere, paraboloid, etc.
- The surface integral of a scalar function $f(x,y,z)$ on a surface S is

  $$\int_S f\,dS = \iint_R f(x(u,v), y(u,v), z(u,v))|\overrightarrow{r}'_u \times \overrightarrow{r}'_v|\,dA,$$

  in which

  $$\overrightarrow{r}'_u(t) = \langle x_u, y_u, z_u\rangle \quad \text{and} \quad \overrightarrow{r}'_v(t) = \langle x_v, y_v, z_v\rangle.$$
- The area of S is

  $$\operatorname{Area}(S) = \iint_R |\overrightarrow{r}'_u \times \overrightarrow{r}'_v|\,dA.$$

- The surface integral of a vector field $\vec{F} = \langle x(u,v), y(u,v), z(u,v)\rangle$ on a surface S is
$$\iint_S \vec{F}\, d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_R \vec{F}(x(u,v), y(u,v), z(u,v))\cdot(\overrightarrow{r}'_u \times \overrightarrow{r}'_v)\, dA.$$
Notice that one has to choose between $\overrightarrow{r}'_u \times \overrightarrow{r}'_v$ and $-\overrightarrow{r}'_u \times \overrightarrow{r}'_v$ so that it is ***consistent with the normal vector*** *$\vec{n}$.*
- Important online questions: 1, 2, 3, 6, 7, 9, 10, 15, 16.

§15.7. Stokes' Theorem.

- Let S be a surface in $\mathbb{R}^3$ with boundary closed curve C and their orientations are consistent by the **right-hand rule**. Then
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\vec{n}\, dS \left(= \iint_S (\nabla\times\vec{F})\cdot d\vec{S}\right).$$
The key observation of Stokes' Theorem is that ***the flux surface integral of the curl in the interior equals circulation line integral of the vector field on the boundary****. Also keep in mind that the curve $c: \vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ here is in $\mathbb{R}^3$.*
- A special case when $\vec{F}$ is conservative, i.e. $\nabla\times\vec{F} = 0$, so the surface integral is always zero and hence *the circulation line integral of a conservative vector field along any closed curve is zero.*
- Important online questions: 1, 2, 3, 4, 7, 8, 9.

§15.8. Divergence Theorem.

- Let S be a closed surface enclosing a solid region D in $\mathbb{R}^3$. Then
$$\left(\iint_S \vec{F}\cdot d\vec{S} =\right) \iint_S \vec{F}\cdot\vec{n}\, dS = \iiint_D \nabla\cdot\vec{F}\, dV.$$
The key observation of Divergence Theorem is that ***the triple integral of the divergence in the interior equals outward flux surface integral of the vector field on the boundary****.*
- A special case when $\vec{F}$ is incompressible, i.e. $\nabla\cdot\vec{F} = 0$, so the triple integral is always zero and hence *the flux surface integral of an incompressible vector field on any closed surface is zero.*
- Important online questions: 1, 3, 5, 8, 9, 10, 12, 14.