NATIONAL SCIENCE FOUNDATION
RESEARCH EXPERIENCE FOR UNDERGRADUATES INSTITUTE
UNIVERSIDADE ESTADUAL DE CAMPINAS – UNICAMP
CAMPINAS, BRAZIL, JULY/2004

REPORT ON THE PROJECT:

SYMMETRY BREAKING IN THE MINIMIZATION OF THE FUNDAMENTAL FREQUENCY OF PERIODIC COMPOSITE MEMBRANES

Participants:

Welington Vieira Assunção – Universidade Estadual Paulista, Rio Claro, SP, Brazil
Anne Carolina Bronzi – Universidade Estadual de Campinas, Campinas, SP, Brazil
Lyubov Chumakova – University of Wisconsin, Madison, WI, USA

Supervisor: Renato H. L. Pedrosa – Universidade Estadual de Campinas, Campinas, SP, Brazil

Abstract. It is shown that configurations with minimal first eigenvalue for the 1–periodic composite membrane in $\mathbb{R}^2$ (the membrane is a strip $\Omega = \mathbb{R} \times [0,1]$) are not necessarily the ones invariant under x–translations, for given basic data, if the period is taken sufficiently large.

1. Introduction

The study of vibrating membranes is a classical subject in mathematical physics, and is at the origin of many important developments in the theory of partial differential equations. We recall that the wave equation in two space variables is

$$\frac{\partial^2 f}{\partial t^2} = \Delta f, \tag{1}$$

where $f(x, y, t)$ depends on the position $(x, y) \in \mathbb{R}^2$ and on $t \in \mathbb{R}$, which is thought of as the time variable. The Laplacian Δ is the operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$, which acts only on the space variables.

A vibrating membrane with homogeneous density distribution is described mathematically by this equation, usually with its rest shape given by a domain $\Omega \subset \mathbb{R}^2$ (open with sufficiently regular boundary $\partial\Omega$), with some initial and boundary conditions given beforehand. The shape of the membrane at each instant is given by the graph of f over Ω. The basic reference for this problem is the book by Courant and Hilbert [2].

Following the original idea of Fourier, one tries to find solutions of the form

$$f(x, y, t) = u(x, y)v(t),$$

(called separation of variables) and it follows that direct substitution into (1) gives that nontrivial solutions must satisfy (at non–zero points)

$$\frac{\Delta u}{u} = \frac{v''}{v}.$$

But since the left–hand side depends only on x and y and the right–hand one only on t, they must be constant. This gives rise to the eigenvalue problem for the Laplacian,

$$-\Delta u = \lambda u \text{ in } \Omega,$$

where the minus sign is introduced so that the possible λ's become non–negative in some important cases (see below).

Usually one also has a boundary condition coming from the original problem. We will consider the case where the membrane is fixed along $\partial\Omega$, i.e. $f(x, y, t) = 0$ for all $(x, y) \in \partial\Omega$ and all $t \in \mathbb{R}$. This gives rise to the same condition for u, called the Dirichlet boundary condition. Thus, one is bound to study the possible solutions of the PDE problem

$$\begin{cases} -\Delta u &= \lambda u \quad \text{in } \Omega \\ \quad u &= 0 \quad \text{in } \partial\Omega \end{cases}.$$

In analogy to the linear algebra concepts, a nontrivial u satisfyng such conditions will be called an eigenfunction of the Laplacian and the associated real number λ an eigenvalue.

Given two continuous functions ϕ, ψ in Ω which go to zero at its boundary, their L^2–inner product is given by

$$\langle \phi, \psi \rangle = \int_\Omega \phi\psi \, dxdy.$$

One then has the classical

Theorem 1 (Spectral Theorem for the Laplacian with Dirichlet boundary condition)**.** *There is an unbounded discrete set of (positive) eigenvalues*

$$(0 <)\, \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$$

and associated set of eigenfunctions u_i, which may be taken mutually orthogonal and with norm equal to 1*, with respect to the L^2–inner product in Ω, which is complete in the sense that any (smooth) function $v : \Omega \to \mathbb{R}$ such that $v = 0$ at $\partial\Omega$ may be written as the (Fourier) series*

$$v = \sum_{i=1}^{\infty} \langle v, u_i \rangle u_i.$$

Remark . In the case of one space variable, i.e., of a vibrating string, the u_i's are given by sine functions, the original Fourier series situation.

The (orthonormal) system of eigenfunctions plays the usual role of an orhonormal basis in finite dimensional linear algebra, when we use orthogonal projection to write a vector in that basis (in that case one has a finite sum of projected vectors). It is also called a *Hilbert basis* for the space of functions in Ω (with Dirichlet boundary condition).

The numbers λ_i are related to the squares of the frequency of the basic solutions (called harmonics) for the vibrating membrane. For thas reason, λ_1, the first eigenvalue, is also called the fundamental frequency or pitch of the membrane.

Lord Rayleigh, in the book The Theory of Sound ([9]), studied the values of λ_1 as a function of Ω, once the area of Ω is fixed. He observed that λ_1 for the round disk $D(r) =$

$\{(x,y) \in \mathbb{R}^2 : \ x^2 + y^2 < r^2\}$ of radius r had, among all domains for which he could calculate λ_1 explicitly, the least value possible. He conjectured that as a property characterizing the round disk. It is worth quoting his exact phrasing of the problem (*op. cit.*, Vol. 1, p. 339):

> *We have seen that the gravest tone of a membrane, whose boundary is approximately circular, is nearly the same as that of a mechanically similar membrane in the form of a circle of the same mean radius or area. If the area of a membrane be given, there must evidently be some form of the boundary for which the pitch (of the principal tone) is the gravest possible, and this can be no other than the circle.*

This was proved to be true in the mid 1920's by Faber and Krahn, independently, now known as the Faber–Krahn inequality:

Theorem 2 (Faber–Krahn inequality)**.** *For all domains $\Omega \subset \mathbb{R}^2$ of a given area $A > 0$, one has*

$$\lambda_1(\Omega) \geq \lambda_1(D(r)),$$

where r is such that $\pi r^2 = A$, and if equality occurs, Ω must be a round disk (maybe centered at some other point).

The result is also true for higher dimensions, with $D(r)$ substituted by the round ball of the given volume.

The spectral theory in Theorem 1 extends in the same form to the case where the mass density of the membrane is a variable (positive) function $\rho(x,y)$. Then, the eigenvalue problem becomes

$$\begin{cases} -\Delta u &= \lambda \rho u \quad \text{in } \Omega \\ \quad u &= 0 \quad \text{in } \partial\Omega \end{cases} .$$

Now, Lord Rayleigh's variational problem for the first eigenvalue acquires a different form. One considers Ω given, then, assuming that the density is bounded between two values, i.e., $0 < \rho_1 \leq \rho \leq \rho_2 < \infty$, one fixes the total mass

$$M = \int_\Omega \rho \, dx dy,$$

and asks which mass distribution ρ satisfying these conditions gives the least λ_1 possible. The result is a bit surprising, since ρ becomes discontinuous. Let χ_A be the characteristic function of $A \subset \Omega$, i.e.,

$$\chi_A(x,y) = \begin{cases} 1 & \text{if } \ (x,y) \in A \\ 0 & \text{if } \ (x,y) \in A^c (= \Omega \setminus A) \end{cases} ,$$

Theorem 3. *The distribution function ρ which minimizes λ_1 is given by*

$$\rho = \rho_1 \chi_D + \rho_2 \chi_{D^c},$$

where D is a (closed) subset of Ω. More over, if u the first eigenfunction associated to λ_1 for this configuration, D is a sublevel set of u, i.e.,

$$D = \{(x,y) \in \Omega : \ u(x,y) \leq c\},$$

for some $c > 0$.

This result was proved for the 1–dimensional case by Krein ([7]) and for general n by Chanillo *et al* ([1]).

We observe that the condition of fixed total mass becomes a condition on the area of D, since

$$M = \int_\Omega \rho\, dxdy = \rho_1|D| + \rho_2|D^c| = \rho_1|D| + \rho_2(|\Omega| - |D|),$$

where $|\cdot|$ is the area of a set, implies that

$$|D| = \frac{\rho_2|\Omega| - M}{\rho_2 - \rho_1},$$

and Ω, ρ_1, ρ_2 and M are given.

The resulting (optimal) membrane is what is called a *composite membrane*: it is made of two materials, distinguished by their mass density distributions. So, it is possible to turn this situation around, and study vibrating membranes having this type of ditribution of mass.

One chooses Ω and $D \subset \Omega$, assumes D has mass density distribution ρ_1, its complement in Ω has mass density distribution ρ_2, then arrives at the following Dirichlet boundary eigenvalue problem:

$$(*)_{\alpha,D} \begin{cases} -\Delta u + \alpha\chi_D u &= \lambda u \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \partial\Omega \end{cases},$$

where $\alpha > 0$ is a constant which depends on ρ_1, ρ_2 and the area of D (or the total mass). We will consider the variational problem for the first eigenvalue of this problem.

2. The variational problem for the first eigenvalue of a composite membrane

First of all, again there is a spectral theorem as Theorem 1 for problem $(*)_{\alpha,D}$, resulting in a sequence $(0 <)\,\lambda_1 < \lambda_2 \leq \dots$ of eigenvalues, with associated (normalized) eigenfunctions u_i. But observe that u_i cannot be C^2, since the Δu is not continuous (just move the term $\alpha\chi_D u$ to the right in the above equation). This is an interesting feature of this problem, unlike most elliptic problems, where, usually, solutions tend to be very regular (mostly, analytic). Also, the first eigenfunction (u_1, associated to λ_1) cannot change sign, by a general result for such problems (maximum principle), and we will consider the positive normalized one (just change its sign if necessary).

We can now introduce the variational problem for $\lambda(D) = \lambda_{\Omega,\alpha}(D)$, which is the way we will denote the first eigenvalue for the problem $(*)_{\alpha,D}$ stated at the end of the previous section, depending on D.

Variational problem for $\lambda_{\Omega,\alpha}(D)$: *fix Ω and $\alpha > 0$. For each $\delta \in (0,1)$, minimize $\lambda_\Omega(\alpha, D)$ among all $D \subset \Omega$ such that $|D| = \delta|\Omega|$.*

Choosing $\delta \in (0,1)$ is the same as choosing the area of D, so we are, in fact, minimizing $\lambda_{\Omega,\alpha}(D)$ among all subsets of Ω with some fixed area. We will also mention the first normalized (positive) eigenfunction $u = u_1$ and call the pair (u, D) an *optimal pair* for the above problem and (Ω, D) an *optimal α–composite membrane for the volume $\delta|\Omega|$ in Ω.*

Before stating some basic properties of solutions, it is important to remark that minimizers for this problem and for the original problem for the variable distribution problem are the same, by a result of [1]. There are many relevant known results about a minimizing pair (u, D). We include some of them here (the basic reference is the [1], but see also [7, 4, 3]):

(1) ***Existence and regularity***: for any $\alpha > 0$ and $\delta \in [0, 1]$ there exists an optimal pair (u, D) for α, δ in Ω. Moreover, it satisfies:
  (a) $u \in C^{1,\sigma}(\Omega) \cap H^2(\Omega) \cap C^\gamma(\overline{\Omega})$ for some $\gamma > 0$ and every $\sigma < 1$;
  (b) D is a sublevel of u, i.e. there is a $c \geq 0$ such that $D = \{u \leq c\}$;

(2) ***Symmetry with convexity***: if Ω is symmetric and convex with respect to a line l, then an optimal pair (u, D) is symmetric and D^c is convex with respect to L. (Ω is convex with respect to a line L if the intersection of Ω with every line l perpendicular to L is an interval in l.) **Special case**: if Ω is a round disk, solutions are circularly symmetric (D is an annulus attached to the boundary of the disk).

(3) ***Symmetry breaking for 2–dim'l annuli***: let

$$\Omega = \{x \in \mathbb{R}^2 : \ a < \|x\| < a + 1\}.$$

For any α, $\delta > 0$ there is a constant $a_{\alpha,\delta} > 0$, depending only on α and δ, such that, if $a \geq a_{\alpha,\delta}$, minimizers D, with $|D|/|\Omega| = \delta$, are not circular (see Figure 1).

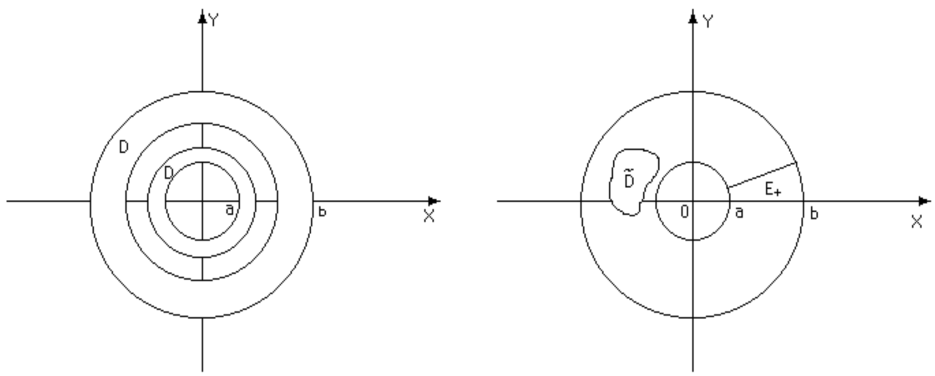

FIGURE 1. 2–dimensional annulus: symmetric D and test domain $\tilde{D}$

Remarks . (1) Properties 1. and 2. are valid for $\mathbb{R}^n$, all n (*cf.* [1]). The symmetry–breaking for 2–dimensional annuli is also from [1]. For higher dimensional annuli see [8].
(2) Numerical examples showing the symmetry–breaking phenomenon for 2–dimensional annuli may be found in [1].

3. Symmetry–breaking of optimal 2–dimensional periodic composite membranes

The purpose of this project is to study symmetry properties of optimal composite membranes in a 2–dimensional strip

$$\Omega_{a,b} = \{(x, y) \in \mathbb{R}^2 : \ a < y < b\},$$

where $a < b$ are real numbers, but restricted to periodic configurations only (in the x direction). In other words, all the eigenfunctions and D considered will be taken of some given period T in the x variable. Since the width $b - a$ is arbitrary, we fix it at 1 and take $a = 0$, $b = 1$ (see below, after the statement of Theorem 4, on the dependence of the result on the width value).

Even though Ω is not bounded, once one fixes the period T, it is possible to consider the piece of right cylinder

$$x^2 + (z - R)^2 = R^2, \quad 0 < y < 1, \quad \text{where } R = \frac{T}{2\pi},$$

as our domain Ω (an open subset of the cylinder $x^2 + (z - R)^2 = R^2$ in $\mathbb{R}^3$) (see the Figure 2 below).

The development the whole theory of periodic composite membranes is beyond the objectives and scope of the project, but we observe that all the general properties of minimizer given for the bounded case, 1. (existence and regularity) and 2. (symmetry in convex case), should also hold in this case. Property 2. implies symmetries with respect to a segment given $x = x_0$ and with respect to the line $y = 1/2$, with similar proofs. We will concentrate in proving an anologous result to the symmetry–breaking property 3. above, which we state now.

In this case the total area of Ω should be thought of as that of one period, i.e., the area of the piece of the cylinder as above, given by $|\Omega| = T = 2\pi R$.

The result we will prove is the following:

Theorem 4. *For each $\alpha > 0$ and $\delta \in (0, 1)$, there is a constant $T_{\alpha,\delta} > 0$, depending only on α and δ, such that, if $T \geq T_{\alpha,\delta}$, a minimizing D in Ω with area $|D| = \delta|\Omega|$ is not invariant under x–translations (or, equivalently, is not invariant under rotations of the cylinder).*

Remark . If one vary the width of the strip, the constant will also depend on it, but it will be clear that it only introduces a correction. The result is essentially the same, and putting it equal to 1 eliminates carrying this correction around.

4. Proof of Theorem 4

The proof is inspired by the one in [1] for the 2–dimensional annulus. Let

$$\Omega = \{(x, y) \in \mathbb{R}^2 : \ y \in (0, 1)\}.$$

Instead of working on Ω with periodic data and functions, we will fix $T > 0$ and let Ω_T be the fundamental domain

$$\Omega_T = \{(x, y) \in R^2 : \ x \in [0, T], \ y \in (0, 1)\},$$

and consider data and functions which coincide at $x = 0$ and $x = T$.

A $D \subset \Omega_T$ which is x–independent (i.e., for which the periodic extension in the x direction is invariant under x–translations) may be written in the form

$$D = \{(x, y) \in \Omega_T : \ y \in D_1 \subset (0, 1)\}.$$

Observe that the relevant boundary of Ω_T will be only the segments $y = 0$ and $y = 1$. The segments $x = 0$ and $x = T$ are thought of, in fact, as interior points.

Now let u be the first eigenfunction for D, with eigenvalue σ, for the composite membrane problem:

$$(*)\begin{cases} -\Delta u + \alpha\chi_D u &= \sigma u \quad \text{on } \Omega_T \\ u(x,0) &= u(x,1) = 0 \\ u(0,y) &= u(T,y) \end{cases}.$$

For T sufficiently large (depending on α and $\delta = |D|/|\Omega|$) we will construct a comparison domain $\tilde{D}$ and a function $\tilde{u}$ which satisfy

$$\frac{\int_{\Omega_T} |\nabla \tilde{u}|^2 + \alpha \int_{\Omega_T} \chi_{\tilde{D}} \tilde{u}^2}{\int_{\Omega_T} \tilde{u}^2} < \sigma. \tag{2}$$

This shows that D is not an optimal configuration and hence implies the theorem.

In order to construct $\tilde{D}$ and $\tilde{u}$, first pick $N = N(\delta)$ with

$$\delta < 1 - \frac{1}{2N}$$

and consider the piece of Ω_T given by

$$E_N = \Omega_T \cap \{(x,y) : \ 0 \le x \le T/2N\}.$$

Then let $\tilde{u}$ be the first Dirichlet eigenfunction of the Laplacian on E_N,

$$\begin{aligned} -\Delta \tilde{u} &= \lambda_1(E_N) \quad \text{on } E_N, \\ \tilde{u} &= 0 \quad \text{on} \quad \partial E_N \end{aligned} \tag{3}$$

extended to zero on $\Omega \setminus E_N$, and $\lambda(E_N)$ be the first eigenvalue.

Let $\tilde{D}$ be any (closed) subset of $\Omega \setminus E_N$ with $|\tilde{D}| = |D|$. This is possible since $|D|/|\Omega| = \delta < 1 - 1/2N = |\Omega \setminus E_N|/|\Omega|$ (see Figure 2).

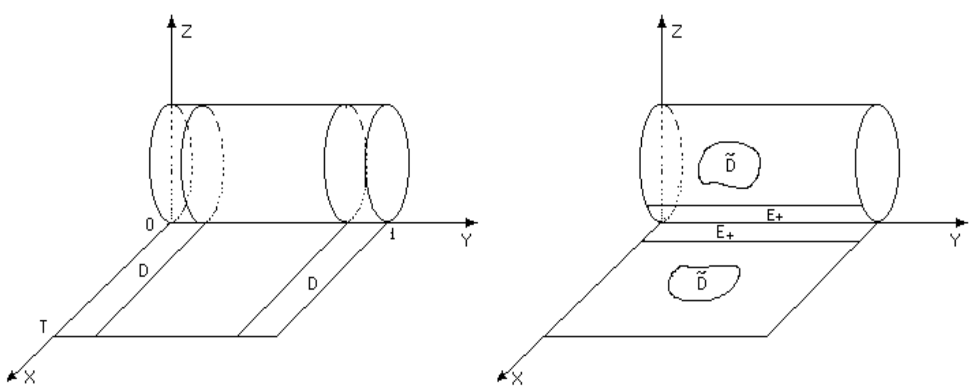

FIGURE 2. Periodic case: symmetric D and test domain $\tilde{D}$

Note that since $\operatorname{supp} \tilde{u} \cap \tilde{D} = \emptyset$, we have

$$\frac{\int_{\Omega_T} |\nabla \tilde{u}|^2 + \alpha \int_{\Omega_T} \chi_{\tilde{D}} \tilde{u}^2}{\int_{\Omega_T} \tilde{u}^2} = \frac{\int_{E_N} |\nabla \tilde{u}|^2}{\int_{E_N} \tilde{u}^2} = \lambda_1(E_N),$$

so (2) is equivalent to

$$\lambda_1(E_N) < \sigma. \tag{4}$$

In order to prove this, we need to introduce a third eigenvalue problem, which is intermediate between (∗) and (3).

We will consider the class of solutions of type

$$v(x,y) = h(y)\sin\left(\frac{2N\pi x}{T}\right), \tag{5}$$

and will define v to be the lowest eigenfunction for the problem (∗) among functions of this type. Let τ be the associated eigenvalue. Note that problem (∗) for such functions is equivalent to the problem

$$\begin{aligned} -h''(y) + \left(\frac{2N\pi}{T}\right)^2 h(y) + \alpha\chi_{D_1}(y)h(y) &= \tau h(y) \quad \text{on } y \in [0,1] \\ h(0) = h(1) &= 0 \end{aligned} \tag{6}$$

for h. Thus, h is the first eigenfunction of this Sturm-Liouville problem, and the eigenvalue τ is characterized by

$$\tau = \inf_{g\in S} \frac{\int_0^1 \left\{(g')^2 + \left[\alpha\chi_{D_1} + \left(\frac{2N\pi}{T}\right)^2\right] g^2\right\} dy}{\int_0^1 g^2 dy}, \tag{7}$$

where $S = \{g \in C^1[0,1] : \ g(0) = g(1) = 0\}$.

From this the (well-known) fact that h does not change sign on $[0,1]$ is evident; so we may assume

$$h \geq 0.$$

We will compare u with v and v with $\tilde{u}$. The following lemmata provide the needed estimates.

Lemma 1. *Let σ be the lowest eigenvalue for the problem (∗) on $\Omega_T = \{(x,y) \in R^2 : x \in (0,T),\ y \in (0,1)\}$, and let τ be the lowest eigenvalue for eigenfunctions of the form $v(x,y) = h(y)\sin(\frac{2N\pi x}{T})$ on Ω_T. Then we have*

$$\tau - \sigma \leq \left(\frac{2N\pi}{T}\right)^2. \tag{8}$$

Proof. Since χ_D is assumed independent of x, a simple calculation with separate variables and the fact that the first eigenfunction of (∗) does not change sign implies that the first eigenfunction of (∗) with D independent of x is also x–independent: $u = f(y)$. Now consider the trial function $\omega(x,y) = f(y)\sin(\frac{2N\pi x}{T})$. We have

$$\tau \leq \frac{\int_{\Omega_T}(|\nabla\omega|^2 + \alpha\chi_D\omega^2)}{\int_{\Omega_T}\omega^2}.$$

Thus,

$$\tau \leq \frac{\int_0^1((f'(y))^2 + \left(\frac{2N\pi}{T}\right)^2 f(y)^2 + \alpha\chi_{D_1} f(y)^2)dy}{\int_0^1 f(y)^2 dy} = $$
$$= \frac{\int_0^1((f'(y))^2 + \alpha\chi_{D_1} f(y)^2)dy}{\int_0^1 f(y)^2 dy} + \frac{\int_0^1 \left(\frac{2N\pi}{T}\right)^2 f(y)^2 dy}{\int_0^1 f(y)^2 dy}$$

By definition of $f(y)$ we get

$$\tau \leq \sigma + \frac{\int_0^1 \left(\frac{2N\pi}{T}\right)^2 f(y)^2 dy}{\int_0^1 f(y)^2 dy} \leq \sigma + \left(\frac{2N\pi}{T}\right)^2.$$

The claim follows. □

Lemma 2. *Define v as above. Assume D is radial and $|D|/|\Omega| = \delta$. There exists a positive constant $c_{\alpha,\delta}$, independent of T, such that for all $T \geq 1$ we have*

$$\frac{\int_D v^2}{\int_\Omega v^2} \geq c_{\alpha,\delta}.$$

Proof. We see from $v(x,y) = h(y)\sin(\frac{2N\pi x}{T})$ that

$$\frac{\int_D v^2}{\int_\Omega v^2} = \frac{\int_0^1 \chi_{D_1}(y)h(y)^2 dy}{\int_0^1 h(y)^2 dy}, \tag{9}$$

where h satisfies Eq. (6). For τ one has a uniform bound $\tau \leq C_{\alpha,\delta}$ with $C_{\alpha,\delta}$ independent of $T \geq 1$, because from (7) one gets

$$\tau \leq \inf_{g\in S} \frac{\int_0^1 (g')^2 dy}{\int_0^1 g^2 dy} + \alpha + (2N\pi)^2$$

and by using for g any test function on $[0,1]$ one sees that the first term on the right is bounded by some absolute constant.

Therefore, the coefficients of Eq.(6) are uniformly bounded for $T \geq 1$. Also, we have $h \geq 0$. Lemma 3 in appendix then implies that one has

$$\inf_{[\delta/4, 1-\delta/4]} h \geq d_{\alpha,\delta}\|h\|_{L^2(0,1)}.$$

Since $|D_1| = \delta$, we have $|[\delta/4, 1-\delta/4] \cap D_1| \geq \delta/2$. Therefore,

$$\int_0^1 \chi_{D_1}(y)h(y)^2 dy \geq \frac{\delta}{2} \inf_{[\delta/4,1-\delta/4]} h^2.$$

Then we have

$$\frac{\int_0^1 \chi_{D_1}(y)h(y)^2 dy}{\int_0^1 h(y)^2 dy} \geq \frac{\frac{\delta}{2}\inf_{[\delta/4,1-\delta/4]} h^2}{\int_0^1 h(y)^2 dy} \geq \frac{\frac{\delta}{2}(d_{\alpha,\delta}\|h\|_{L^2(0,1)})^2}{(\|h\|_{L^2(0,1)})^2},$$

therefore the lemma is proved. □

End of proof of Theorem 4. We have

$$\tau = \frac{\int_{\Omega_T} |\nabla \tilde{v}|^2}{\int_{\Omega_T} v^2} + \frac{\alpha \int_{\Omega_T} \chi_D v^2}{\int_{\Omega_T} v^2}. \tag{10}$$

Since $v(x,y) = h(y)\sin(\frac{2Nx\pi}{T})$, v vanishes on the segments $x = 0$ and $x = T/2N$. Since $|v|$ and $|\nabla v|$ are periodic in x of period $T/2N$, we can replace Ω by E_N in the first quotient. Therefore, we can use v as test function in the Rayleigh quotient for the Dirichlet Laplacian on E_N and obtain

$$\frac{\int_{\Omega_T} |\nabla \tilde{v}|^2}{\int_{\Omega_T} v^2} = \frac{\int_{E_N} |\nabla \tilde{v}|^2}{\int_{E_N} v^2} \geq \lambda_1(E_N). \tag{11}$$

Combining this with Lemma 2 we therefore get

$$\tau \geq \lambda_1(E_N) + \alpha c_{\alpha,\delta} - \left(\frac{2N\pi}{T}\right)^2.$$

From Lemma 1 we then get

$$\sigma > \tau - \left(\frac{2N\pi}{T}\right)^2 \geq \lambda_1(E_N) + \alpha c_{\alpha,\delta} - \left(\frac{2N\pi}{T}\right)^2.$$

If T is chosen so large that $\left(\frac{2N\pi}{T}\right)^2 \leq \alpha c_{\alpha,\delta}$ gives

$$\sigma > \lambda_1(E_N),$$

and hence the theorem.

5. Appendix: Basic elliptic estimates (reproduced from [1])

Here we collect some well-known facts about uniform estimates for solutions of elliptic equations. We will state these for an equation

$$Pu = 0, \quad P = \Delta + \sum_{j=1}^{n} b_j(x)\frac{\partial}{\partial x_j} + c(x), \quad x \in G, \tag{12}$$

where P has measurable, uniformly bounded coefficients, $u \in C^1(G) \cap C^0(\overline{G})$, and $G \subset R^n$ is a bounded open set. In the following estimates, saying that the constants depend on P will mean that they depend on $\sup_G(b_1, ..., b_n, c)$ and stay bounded when this quantity stays bounded.

First, we have the uniform bound (see [5], Thm. 8.15 and 8.38)

$$\sup_G |u| \leq C_{G,P}(\|u\|_{L^2(G)} + \sup_{\partial G} |u|). \tag{13}$$

Second, we have Harnack's inequality: If $u \geq 0$ on G and G' is a compact subset of G then

$$\sup_{G'} u \leq c_{G,G',P} \inf_{G'} u \tag{14}$$

Combining these two we get the following estimate. For $\epsilon \geq 0$ let $G_\epsilon = \{x \in G : \text{dist}(x, \partial G) \leq \epsilon\}$.

Lemma 3. *For any $\epsilon \geq 0$ there is a positive constant $c_{G,P,\epsilon}$ such that for any $u \in C^1(G) \cap C^0(\overline{G})$ that solves $Pu = 0$ and satisfies $u \geq 0$ one has*

$$\inf_{G_\epsilon} u \geq c_{G,P,\epsilon}(\|u\|_{L^2(G)} - \sup_{\partial G} u). \tag{15}$$

Here we set $\inf_\emptyset u := \infty$.

Proof. We have

$$\begin{aligned} \|u\|^2_{L^2(G)} &= \int_G u^2 = \int_{G_\epsilon} u^2 + \int_{G \setminus G_\epsilon} u^2 \\ &\leq |G_\epsilon| \sup_{G_\epsilon} u^2 + |G \setminus G_\epsilon| \sup_G u^2 \\ &\leq (|G_\epsilon|^{1/2} \sup_{G_\epsilon} u + |G \setminus G_\epsilon|^{1/2} \sup_G u)^2. \end{aligned}$$

Thus,

$$\begin{aligned} \|u\|_{L^2(G)} &\leq (|G_\epsilon|^{1/2} \sup_{G_\epsilon} u + |G \setminus G_\epsilon|^{1/2} \sup_G u) \\ &\leq C_{G,P,\epsilon} \inf_{G_\epsilon} u + |G \setminus G_\epsilon| C'_{G,P}(\|u\|_{L^2(G)} + \sup_{\partial G} u), \end{aligned}$$

where we used Harnack's inequality and the uniform estimate (13). If ϵ is so small that $|G \setminus G_\epsilon| C'_{G,P} < 1/2$ then we can subtract the last two terms, and the claim follows. The claim for larger ϵ then follows from the fact that $\inf_{G_{\epsilon'}} u \geq \inf_{G_\epsilon} u$ if $\epsilon' \geq \epsilon$. ☐

6. Acknowledgements

This is the final report of work developed during the Research Experience for Undergraduates (REU) Institute hosted by the Universidade Estadual de Campinas (UNICAMP), São Paulo, Brazil, in July of 2004. The REU was funded by the National Science Foundation (grant INT 0306998) and FAEP (Unicamp) (grant 019/04).

The authors would like to thank the Department of Mathematics at UNICAMP for their hospitality, their advisor Professor Renato Pedrosa, for suggesting the problem and for useful discussions, and Professors Marcelo Firer and M. Helena Noronha for organizing the event.

References

[1] S. Chanillo, D. Grieser, M. Imai, K. Kurata, I. Ohnishi, *Symmetry breaking and other phenomena in the optimization of eigenvalues of composite membranes.* Commun. Math. Phys. **214** (2000), 315–337.

[2] R. Courant, D. Hilbert, *Methods of Mathematical Physics, v.I*, Interscience, J. Wiley, NY, 1953

[3] S. J. Cox, *The two phase drum with the deepest bass note*, Japan J. Indust. Appl. Math **8** (1991), 345–355.

[4] S. J. Cox, J. R. McLaughlin, *Extremal eigenvalue problems for composite membranes I, II*, Appl. Math. Optim. **22** (1990), 153–167, 169–187.

[5] D. Gilbarg, N.T. Trudinger, *"Elliptic Partial Differential Equations of Second Order"*, Springer-Verlag, New York/Berlin, 1983.

[6] B. Kawohl, *Rearrangements and convexity of level sets in PDE's*, LNM 1150, Springer, 1985.

[7] M. G. Krein, *On certain problems on the maximum and minimum of characteristic values and on Lyapunov zones of stability*, AMS Translations Ser. **2** (1955), 1, 163–187.

[8] R. H. L. Pedrosa, *Minimal partial rearrangements with applications to symmetry properties of optimal composite membranes.* Preprint, 2004.

[9] J. W. S. Rayleigh, *The Theory of Sound*, Vols. 1, 2, Dover Publications, New York, 1945 (unabridged republication of the 2^{nd} revised editions of 1894 (Vol. 1) and 1986 (Vol. 2), Macmillan).

Addresses:

Welington Vieira Assunção
Universidade Estadual Paulista, Rio Claro, SP, Brazil
wass@rc.unesp.br

Anne Carolina Bronzi
Universidade Estadual de campinas, Campinas, SP, Brazil
a015513@dac.unicamp.br

Lyubov Chumakova
University of Wisconsin, Madison, WI, USA
lchumakova@wisc.edu / lchumakova@hotmail.com

Renato H. L. Pedrosa
IMECC CP6065
Universidade Estadual de Campinas
13083-970 Campinas, SP, Brazil
pedrosa@ime.unicamp.br