Parallels in Hyperbolic Geometry

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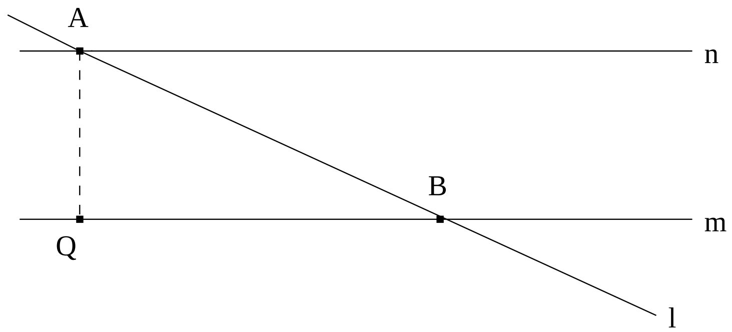
1 Introduction

In Euclidean Geometry, it is well known that when you reflect a point through two concurrent and different lines, you end up with a rotation. In the same way, when you reflect a point through two parallel lines, you end up with a translation. It is interesting to observe that under certain conditions a family of rotations turn into a translation. In fact,

Theorem 1. *Let n and m be two parallel lines in Euclidean Space. Let $A \in n$ and let point Q be the foot of the perpendicular line to m dropped from A. Consider point B on m and let l be the line through A and B.*



Let S_l and S_m be reflections of an arbitrary point P in the plane through lines l and m respectively. The composition of S_l and S_m obtains a rotation centered at B, i.e.,

$$R_B = S_m \cdot S_l$$

If we move point B along line m to infinity, then the line l will eventually coincide with line n. Therefore, the rotation of point P, will become a translation of point P.

In other words,

$$lim_{B\to\infty} S_m \cdot S_l = S_m \cdot S_n$$

We have a proof of the above theorem in Section 2. You should observe that in our proof, we used the Euclidean parallel postulate which states that for every line l and every point P not lying on l, there exists a unique line through P that is parallel to l.

The question is now, can we make the same generalization in hyperbolic geometry? We have to be careful how we answer this question because in hyperbolic geometry the Euclidean parallel postulate does not hold. In fact, given a line in a plane and a point not on the line, we have infinitely many parallels to the line through the point. Never the less, the results still can be generalized in the Hyperbolic case in the context that we explain in section 3.

2 Euclidean Case

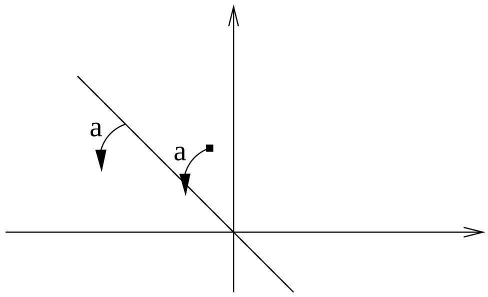
First we need to find the composition of reflections $S_m \cdot S_l$, by setting up a coordinate system with the origin at point Q. Let the line m be the X-axis and $\overline{AQ}$ lie along the Y-axis.

Let point A and B be two fixed points with coordinates, $A = (0, a)$ and $B = (b, 0)$. Let line $l = mx + c$ go through the points A and B. Label the angle between line l and the X-axis as θ and denote $\alpha = 180 - \theta$. Now let point $P = (x, y)$ be an arbitrary point in the coordinate system. Note, we shall write the compositions of reflections in terms of P.

In order to get the rotation R_B, we must first reflect point P through line l and then reflect point P through line m.

2.0.1 Sub-Proof

To reflect point P through the line whose equation is $y = mx + c$ is complex. However, translating l to the origin and then from the origin rotating l onto the X-axis makes the reflection through the line $l = mx + c = 0$ much easier. So then to reflect point P through line l, a composition of translation (T), a

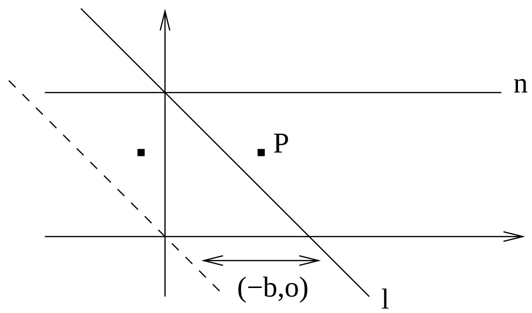


rotation (R_P),then a reflection through line l (S'_l), the inverse rotation (R^{-1}), and finally the inverse translation (T^{-1}) is needed. Therefore, the reflection of P through line l is:

$$S_l = T^{-1} \cdot R^{-1} \cdot S'_l \cdot R \cdot T$$

To reflect P through l we can follow the next steps:

Step 1: Translate horizontally the line l to the origin, subtracting $\begin{bmatrix} b \\ 0 \end{bmatrix}$



Step 2: Rotate line l around the origin by an angle α using the following rotation matrix

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

Step 3: So we can reflect P through l using the reflection matrix through the X axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Step 4: Now perform an inverse rotation on line l, using the rotation matrix with an angle $-\alpha$

Step 5: Finally translate the line l by adding $\begin{bmatrix} b \\ 0 \end{bmatrix}$.

If we consider only the composition of the two rotations and the reflection, we obtain this multiplication of matrices:

$$R^{-1} \cdot S_l' \cdot R = \begin{bmatrix} \cos -\alpha & -\sin -\alpha \\ \sin -\alpha & \cos -\alpha \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}.$$

Manipulating this product of matrix, we have:

$$\begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}.$$

$$\begin{bmatrix} \cos \alpha . \cos \alpha - \sin \alpha . \sin \alpha & -\sin \alpha . \cos \alpha - \sin \alpha . \cos \alpha \\ -\sin \alpha . \cos \alpha - \sin \alpha . \cos \alpha & -\cos \alpha . \cos \alpha + \sin \alpha . \sin \alpha \end{bmatrix}$$

Recall that $\cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha$ and $\sin(2\alpha) = 2 \sin \alpha . \cos \alpha$, so we have:

$$\begin{bmatrix} \cos(2\alpha) & -\sin(2\alpha) \\ -\sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$$

Considering the translations, we have a final formula to a reflection through a generic line l whose inclination is equal θ:

$$P' = T^{-1} \cdot R^{-1} \cdot S_l' \cdot R \cdot T \begin{bmatrix} \cos(2\alpha) & -\sin(2\alpha) \\ -\sin(2\alpha) & -\cos(2\alpha) \end{bmatrix} \cdot (P - \begin{bmatrix} b \\ 0 \end{bmatrix}) + \begin{bmatrix} b \\ 0 \end{bmatrix} \quad (1)$$

Reflecting P' through X-axis and using formula (1):

$$P'' = S_m \cdot T^{-1} \cdot R^{-1} \cdot S_l' \cdot R \cdot T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot P'$$

$$P'' = \begin{bmatrix} \cos(2\alpha) & -\sin(2\alpha) \\ \sin(2\alpha) & \cos(2\alpha) \end{bmatrix} \cdot (P - \begin{bmatrix} b \\ 0 \end{bmatrix}) + \begin{bmatrix} b \\ 0 \end{bmatrix} \quad (2)$$

$$P'' - \begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(2\alpha) & -\sin(2\alpha) \\ \sin(2\alpha) & \cos(2\alpha) \end{bmatrix} \cdot (P - \begin{bmatrix} b \\ 0 \end{bmatrix})$$

This composition of reflections results in a rotation by the angle 2α centered in $B = (b, 0)$.

Solving the product of formula (2), we have:

$$P'' = \begin{bmatrix} (\cos 2\alpha).(x-b) - (\sin 2\alpha).y + b \\ (\sin 2\alpha).(x-b) + (\cos 2\alpha).y \end{bmatrix} \tag{3}$$

Using the relations about $\sin\alpha$ and $\cos\alpha$:

$$\sin\alpha = \frac{a}{\sqrt{a^2+b^2}}$$
$$\cos\alpha = \frac{b}{\sqrt{a^2+b^2}}$$

Now, we want to describe $\sin 2\alpha$ and $\cos 2\alpha$ in terms of a and b:

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = \frac{2ab}{a^2+b^2}$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \frac{b^2-a^2}{a^2+b^2}$$

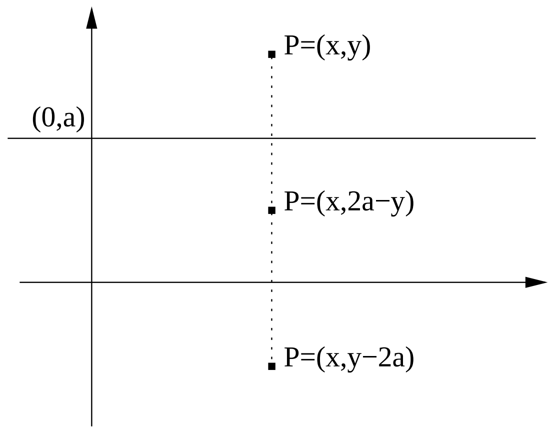
Using this equalities in (3), we obtain:

$$P'' = \begin{bmatrix} x.\frac{b^2-a^2}{a^2+b^2} - b\frac{b^2-a^2}{a^2+b^2} - \frac{2aby}{a^2+b^2} + b \\ \frac{2abx}{a^2+b^2} - \frac{2ab^2}{a^2+b^2} + y\frac{b^2-a^2}{a^2+b^2} \end{bmatrix} \tag{4}$$

Now, that the matrix is in terms of b and a, the limit as b goes to infinity can be taken:

$$\lim_{b\to\infty} P'' = \lim_{b\to\infty} \begin{bmatrix} x.\frac{b^2-a^2}{a^2+b^2} - b\frac{b^2-a^2}{a^2+b^2} - \frac{2aby}{a^2+b^2} + b \\ \frac{2abx}{a^2+b^2} - \frac{2ab^2}{a^2+b^2} + y\frac{b^2-a^2}{a^2+b^2} \end{bmatrix} = \begin{bmatrix} x \\ y-2a \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ 2a \end{bmatrix}$$

Therefore, we have shown, thus far, that as b goes to infinity, the rotation of the point P becomes a translation of point P.



This transformation happens because line l converges to line n. The following is a formal explanation.

2.1 Proof $l = n$

With the same coordinate system as before, let $B = (b, 0)$ go to infinity and $A = (0, a)$ remains constant.

$$l : y = mx + c$$

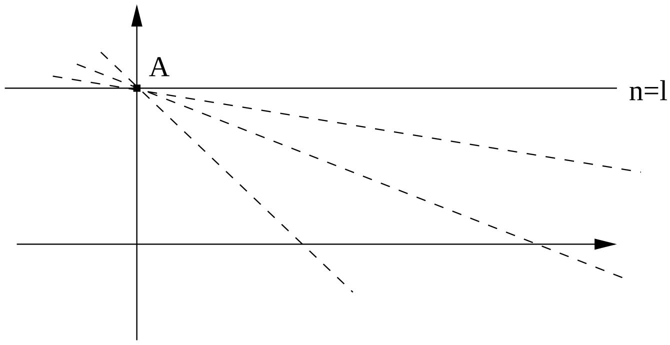
$$m = \tan\theta = -\tan\alpha = \frac{-a}{b}$$

$$\lim_{b\to\infty} \frac{-a}{b} = 0$$

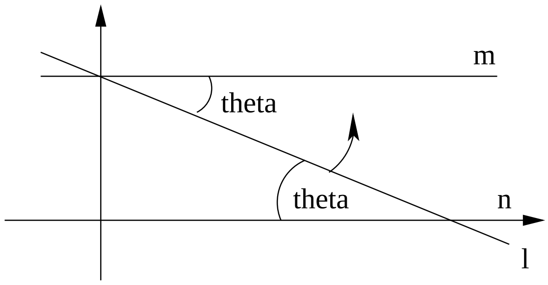
But l passes through the point $A = (0, a)$, so:

$$a = m0 + c \Rightarrow c = a$$

Therefore, as B goes to infinity the equation of l is $y = a$, which equals to the equation of line n.



Notice, in the Euclidean case we showed how line l converges to line n by first defining a coordinate system. In the coordinate system we are assuming the Euclidean Parallel Postulate. However, we could have easily shown that line l converges to line n by using the converse of the Alternate Interior Angle Theorem. Observe in the below picture that θ becomes smaller as B moves along line n and since line n and m are parallel and cut by a transversal l we can use the converse of the Alternate Interior Angle Theorem to conclude that the angle between line m and l equals θ. So, this angle also becomes smaller. In fact, θ converges to zero. At this point the transversal l will converge to line n. Note that this proof uses the the converse of the Alternate Interior Angle Theorem which also assumes the Euclidean Parallel Postulate. This brings us to the conclusion that the Euclidean proof can not be directly used to generalize results in the Hyperbolic case.

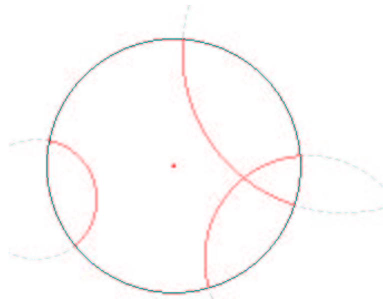


What we showed in tha Euclidean case is that two reflections, respectively, through the transversal and one of the parallel lines starts as rotation, but as the intersection moves along the same parallel line to infinite, the rotation becomes just a translation. The transformation of the rotation into a translation shows what is happening to the transversl as you move the point of intersection to infinite in the Euclidean plane. The transversal converges to the other parallel line. Now, does this same conversion happen in the hyperbolic plane? We will explore this question but, first in the next section we state some Hyperbolic properties.

3 Hyperbolic Case

3.1 Properties of Hyperbolic Geometry

In Hyperbolic geometry we can study many models, however we will only describe the hyperbolic plane in the Poincare Disk Model. In this model the circumference of the disk represents infinity. All points and lines exist only inside the disk. Lines in this plane are called geodesics and are defined as arcs of circles that meet the circumference orthogonally.



Lets recall some important facts about hyperbolic geometry:

1. The hyperbolic Parallel Postulate is just the negation of the Euclidean Parallel Postulate.

   Hyperbolic Parallel Postulate: *there exist a line l such that for some point P not on l at least two lines parallel to l pass through P.*

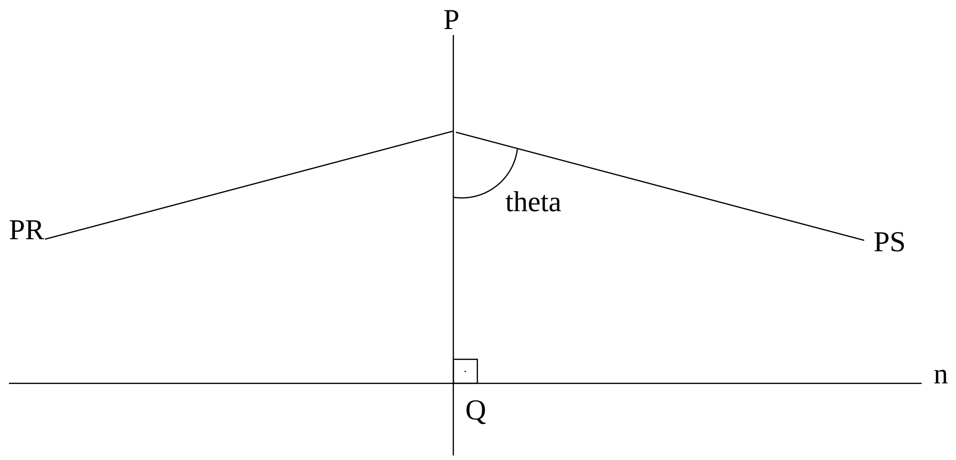
2. We can assume all axioms of neutral geometry, so we can use the following theorem:

   Theorem 2. *Given a line l and a point $P \notin l$, let Q denote the foot of the perpendicular dropped from P to Q. Then there exist two rays $\vec{PR}$ and $\vec{PS}$ on opposite sides of $\vec{PQ}$ such that*

   (a) The ray $\vec{PR}$ and $\vec{PS}$ do not intersect l.

(b) *A ray $\vec{PX}$ intersects line l if and only if $\vec{PX}$ is between $\vec{PR}$ and PS.*

(c) *$\angle QPR \simeq \angle QPS$.*



We will not state the full proof of this theorem however we will give a brief outline ([N]).

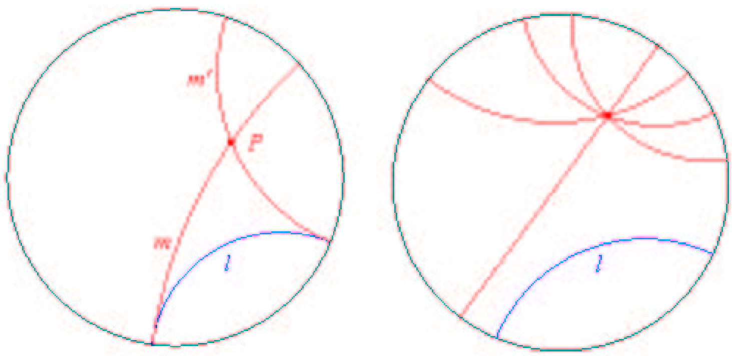
The measurement axioms for angles imply that for every real number $x \in [0, 180]$ there exists a point X on one side of the line through P and Q such that the measure of the angle $\angle XPQ$ equals x. Now, let $Y = \{x \in [0, 180^o]/$ the ray $\vec{PX}$ intersects $l\}$.Note that Y is non-empty and bounded. Therefore, the set has a supremum $supY = s$. Then there exists a point S on one side of PQ such that $m(\angle SPQ) = s$ and that the ray $\vec{PS}$ does not intersect line l. In fact, that in Euclidean geometry the sets supremum will be 90^o and in Hyperbolic geometry the supremum of the set is less than 90^o.

In hyperbolic geometry the measure of this angle is called *the angle of parallelism* of l at P and the rays PR and PS the *limiting parallel rays* for P and l.

3. In Hyperbolic geometry there are infinitely many parallels to a line through a point not on the line. However, there are two parallel lines that contains the limiting parallel rays which are defined as lines *critically parallel* to a line l through a point $P \notin l$. In the Poincare model lines that are critically parallel meet only at infinity.

4. Any line passing through the point P that does not intersect l nor contains the limiting parallel rays are simply parallel to l at point P.

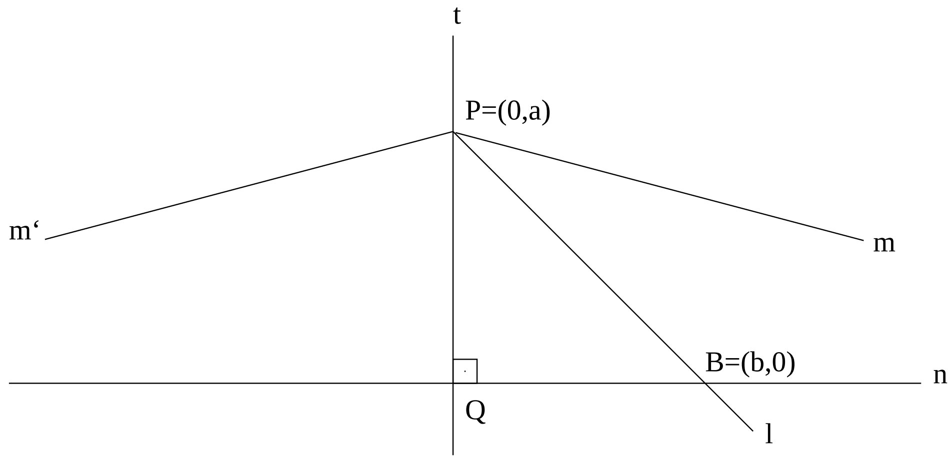
The following illustrations are a visualization of parallel and critically parallel lines in the Poincare model.



3.2 Stating the Theorem

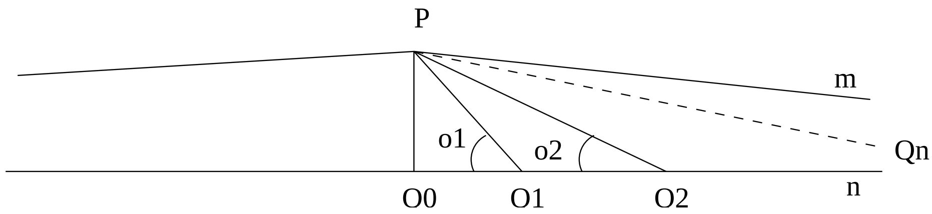
Theorem 3. *Let n be a line in the hyperbolic plane. Choose point P in the plane not on the line n. Let Q be the foot of the perpendicular line dropped from P to line n. Denote this line by t. Let m and m' be the two critically parallel lines to n through point P. Choose point $B \in n$ and let l be a line through point P and B. Let 0 be the coordinate of Q and b be the coordinate of B on the line l*

Then as b approaches $+\infty$ i.e. point B moves in the positive direction along line n, then line l tends to one of the two lines critically parallel to n through point P. Likewise, if b approaches $-\infty$ i.e. point B moves in the negative direction along line n, then line l tends to the other line critically parallel to n through point P. (see fig).



In order to prove the above theorem we first need to prove the following lemma.

Lemma 1. *Denote point* Q *as* Q_0 *and choose point* Q_1 *on the line* n *such that* $\overline{PQ_0} = \overline{Q_0Q_1}$*. In the same way, let* Q_2 *be on the line* n *such that* $\overline{PQ_1} = \overline{Q_1Q_2}$*. So we can construct a sequence of points* (Q_n) *such that* $\overline{PQ_{n-1}} = \overline{Q_{n-1}Q_n}$*, for* $n = 1, 2, \ldots$*. Denote* θ_n *as the measure of the angle* $\angle PQ_nQ_{n-1}$*. Then the sequence* (s_n) *given by* $s_n = \sum_{i=1}^{n} \theta_i$ *converges to the parallelism angle.*



Proof: Since $\overline{PQ_{n-1}} = \overline{Q_{n-1}Q_n}$, observe that the triangle $\triangle PQ_{n-1}Q_n$ is isosceles. Therefore, $\angle Q_{n-1}PQ_n \simeq \angle PQ_nQ_{n-1}$ and the measure of the angle $\angle Q_{n-1}PQ_n$ is θ_n.

From hyperbolic geometry we know that the angle sum of a triangle is less than $180°$. So, for the first triangle we have

$$\angle PQ_0Q_1 + \angle Q_0PQ_1 + \angle Q_0Q_1P < 180^o$$

and since $\angle PQ_0Q_1 = 90^o$

$$90^o + \theta_1 + \theta_1 < 180^o \Rightarrow \theta_1 < 45^o$$

Analogously, for the second triangle

$$\angle PQ_1Q_2 + \angle Q_1PQ_2 + \angle Q_1Q_2P < 180^o$$

But the angles $\angle PQ_1Q_2$ and $\angle Q_0Q_1P$ are supplementary and if we denote the angle $\angle PQ_1Q_2$ as β_1 we have

$$\beta_1 + \theta_1 = 180^o \Rightarrow \theta_1 = 180^o - \beta_1$$

Therefore

$$\theta_2 < \frac{180^o - \beta_1}{2} \Rightarrow \theta_2 < \frac{\theta_1}{2} < \frac{45^o}{2}.$$

Now, denoting the angle $\angle PQ_2Q_3$ as β_2 we have

$$\beta_2 + \theta_2 = 180^o \Rightarrow \theta_2 = 180^o - \beta_2$$

and for the triangle $\triangle PQ_2Q_3$ we have

$$\angle PQ_2Q_3 + \angle Q_2PQ_3 + \angle Q_2Q_3P < 180^o,$$

$$\theta_3 < \frac{180^o - \beta_2}{2},$$

$$\theta_3 < \frac{\theta_2}{2} < \frac{\theta_1}{4} < \frac{45^o}{4}.$$

This argument can be repeated as many times as needed and after n times we have

$$\theta_n < \frac{45^o}{2^{n-1}}$$

Furthermore, we know that $\sum_{n=1}^{\infty} \frac{45^o}{2^{n-1}}$ is a geometric series that converges to 90^o. So by the Comparison theorem, we can say that $\sum_{n=1}^{\infty} \theta_n$ converges as well, i.e., the sequence (s_n) given by $s_n = \sum_{i=1}^{n} \theta_i$ converges.

Note, the measurement axioms for angles imply that for every real number $x \in [0, 180]$ there exists a point X on one side of the line through P and Q_0 such that the measure of the angle $\angle XPQ_0$ equals x. Now, let $S = \{x \in [0, 180^o] /$ the ray $\vec{PX}$ intersects $n\}$, where s_n is contained in the set S. Since

(s_n) is an increasing bounded sequence that converges, we know that the limit of this sequence when n goes to infinity is the supremum (sup) of the set S. The sup of this set is the angle of parallelism, therefore, s_n converges to the angle of parallelism.

Proof of Theorem 1: In lemma 1 we constructed a sequence (Q_n) of points on the line n. Now, let us denote the coordinates of these points as q_n where, $q_0 = 0$. Lemma 1 proves that when $q_n \to \infty$ the sequence (s_n) given by $s_n = \sum_{i=1}^{n} \theta_i$ converges, i.e.,

$$\lim_{n\to\infty} s_n = \theta$$

where θ is the angle of parallelism. So, the lemma proves that $\forall\epsilon > 0, \exists N$ such that if $n > N$ then $|s_n - \theta| < \epsilon$.

Let B be any point on n such that its on the same side of the line t that the points $Q_1, Q_2, ..., Q_n, ...$ are constructed. Let b be the coordinate of the point B and $\alpha(b)$ the angle $\angle Q_0PB$. We want to prove that

$$\lim_{b\to\infty} \alpha(b) = \theta$$

i.e. $\forall\epsilon > 0, \exists M$ such that if $b > M$ then $|\alpha(b) - \theta| < \epsilon$.

Given $\epsilon > 0$, let $\epsilon' = \frac{\epsilon}{2}$. Since $\lim_{n\to\infty} s_n = \theta$, we have that for ϵ', $\exists N_1$ such that if $n > N_1$ then $|s_n - \theta| < \epsilon'$.

We can write

$$|\alpha(b) - \theta| = |\alpha(b) - s_n + s_n + \theta| \leq |\alpha(b) - s_n| + |s_n - \theta|$$

As $q_n \to \infty$ and $\forall b$, $0 = q_0 < b$. So we can conclude that when $q \to \infty \exists n$ such that

$$q_n \leq b \leq q_{n+1}, \forall b$$

Since α is increasing, we have

$$\alpha(q_n) = s_n \leq \alpha(b) \leq s_{n+1} = \alpha(q_{n+1})$$

We know that a convergent sequence is a Cauchy sequence, and hence (s_n) is a Cauchy sequence. Therefore, given the same $\epsilon > 0$ and $\epsilon' = \frac{\epsilon}{2}$, we have that $\exists N_2$ such that if $n > N_2$ and $m > N_2$ then $|s_n - s_m| < \epsilon'$. In particular,

$$|s_n - s_{n+1}| < \epsilon' \Rightarrow s_n - \epsilon' < s_{n+1} < s_n + \epsilon'$$

If $n > N_2$, and b is such that $q_n \leq b \leq q_{n+1}$. We have

$$s_n - \epsilon' < s_n \leq \alpha(b) \leq s_{n+1} < s_n + \epsilon' \Rightarrow$$

$$s_n - \epsilon' < \alpha(b) < s_n + \epsilon' \Rightarrow |\alpha(b) - s_n| < \epsilon'$$

Now, take $N = \max\{N_1, N_2\}$. We conclude that if $n > N$ then

$$|\alpha(b) - \theta| \leq |\alpha(b) - s_n| + |s_n - \theta| < \epsilon' + \epsilon' = \epsilon$$

which implies,

$$\lim_{b \to \infty} \alpha(b) = \theta$$

Observe that if $b \to +\infty$ line l converges to line m. In the same way if $b \to -\infty$ line l converges to line m'.

4 Conclusion

According to this theorem, if we have two concurrent lines intersecting at a point, and we move the point of intersection to infinity along one of the two lines, these lines will become critically parallel.

In conclusion, in the Euclidean case, if we have two parallel lines and a transversal and send the intersection point to infinity along one of the two parallel lines, the transversal will become the other parallel line. However, in the Hyperbolic case, the same will occur only when the parallel lines are critically parallel. Otherwise, if the lines are just parallel then the transversal will not converge to the other parallel line.

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