Models and algorithms for hydrothermal scheduling

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Abstract

This paper intends to explore a minimization problem associated with the cost of operating a hydroelectric and thermal power supply system. The problem focuses on a fixed span of time for which a changing market demand must be met. While this problem has obvious relevance to organizations attempting to meet power demands on a daily basis, it is also an instructive illustration of the many mathematical approaches that can be applicable to operations research.

1 Introduction

We consider a simple power grid system consisting of one hydroelectric plant and a number of thermoelectric plants. A hydroelectric plant generates power through release of water from its reservoir into a turbine. We assume that there are no variable costs in this process. A thermoelectric plant generates power by burning coal or some other sort of fuel, which is quite costly, with costs increasing with rising output. The total power produced by all plants together must match market demand. So, given in advance the market's power demand for a certain period of time, what is the cheapest way to schedule power generation to match demand?

This separates immediately into two distinct (though related) problems. First, what is the optimal way to load a particular power demand between the different thermal plants so as to minimize cost. Secondly, given this optimal load sharing and the resulting cost function of producing power with the thermal plants, what is the optimal way to split power generation between the thermal and hydroelectric plants. Said differently, but more precisely, how should the reservoir be managed to minimize total costs.

To focus first on this later question, we will wish to choose q_t, the month's total outflow of water, at each of N_S time points. We will assume herein that $N_S = 48$, representing monthly stages along four years of scheduling. We begin with an initial volume v_0; later volumes are given by the recursion $v_{t+1} = v_t - q_t$. Importantly, the outflow at each unit of time is bounded by some maximum outflow q^+, representing the maximum capacity of the gates. One should note at this point that we are essentially exploring an optimization problem in 48 dimensions, and we will certainly be expounding upon the difficulties encountered in such a space.

In this model, we assume that the power generated by the hydro at month t depends on q_t, as well as the pressure of the water still in the reservoir, which is related to the current volume by a function, $\rho(v)$, determined by the geometry of the reservoir; for our purposes the hydro power generation will be $h_t = \rho(v_t) \cdot q_t$.

Returning to the former problem, we focus on the N_T thermal plants. In our model, each thermo i can produce only up to g_i^+ units of power. At each time step t, the relation between a thermo's power production $g_{i,t}$ and its cost is given by a quadratic function, $\theta_i(g_{i,t}) = \alpha_i g_{i,t}^2$. So, at any given t we must have that the sum of h_t and the various $g_{i,t}$ match market demand d_t.

We now state the problem in its entirety.

Problem 1

$$\min \sum_{t=0}^{NS-1} \sum_{i=1}^{NT} \alpha_i g_{i,t}^2$$

subject to

$$\begin{aligned} 0 &\leq g_{i,t} \leq g_i^+ \\ v_t &\geq v^- \\ 0 &\leq q_t \leq q^+ \\ v_{t+1} &= v_t - q_t \\ h_t &= \rho(v_t) q_t \\ d_t &= h_t + \sum_{i=1}^{NT} g_{i,t} \end{aligned}$$

where $v^-, q^-, q^+, g_i^+, d_t, \alpha_i, v_0$ and ρ are given.

We will address three formulations of this problem. First we shall consider the original formulation described above, where q_t and v_t assume real values and the set of time points is finite. This leads to the division of into two similar subproblems, as was described above. Secondly we shall apply also restrict the allowed values for volume and outflow at each time point to a finite set; from this we derive an algorithm using dynamic programming. Finally, we explore the possibilities of a continuum limit as the interval between time points approaches zero, leading to a problem which can sometimes be solved exactly and for which we have devised a numerical approximation in terms of a simple differential equation integrator.

2 The Standard Formulation

As preciously mentioned, the problem examined breaks naturally into two parts, the first dealing with optimal scheduling of the thermal plants, and the second with the consequential optimal scheduling of the hydroelectric plant.

2.1 Thermal Plant Scheduling

We first attempt to solve Problem 1 as a traditional optimization problem, finding the minimum value of a function f within a domain $D \in R^n$.

Determining the optimum load distribution among the thermal generators should depend directly only on the total load required of them. That is,

Problem 2

$$\min c(g) = \sum_{i=1}^{N_T} \alpha_i g_i^2$$

subject to

$$\begin{aligned} \sum_{i=1}^{N_T} g_i &= G \\ 0 &\leq g_i \leq g_i^+ \end{aligned}$$

Note that the domain of this problem is the intersection between a hyperplane and a box in R^{NT}. This polytope may have up to $2^{N_T-1} - 1$ cells. In a multivariate calculus course, one learns how to use Lagrange multipliers in order to locate solutions of optimization problems similar to this. However, with these methods we must find the points which satisfy regularity conditions for each cell in the domain; those that do are candidate minima only if they are solutions of a certain system. Even in our case, where the restrictions are all linear and thus regularity conditions are automatically satisfied, we would have to solve a different system for each cell, a combinatorial nightmare. Instead, consider Proposition 1, which solves a part of the problem in a streamlined fashion.

Before detailing it, we offer a brief reminder of the definition of strict convexity:

Definition 1. *A function $h : D \to R$ is said to be* strictly convex *if, for all $a, b \in D$, for any $s \in (0, 1)$, $sh(a) + (1 - s)h(b) > h(sa + (1 - s)b)$; that is, if the secant line to h at a, b is strictly above the graph of h in the entire open interval (a, b).*

If a function h is strictly convex and has a derivative h', then h' is strictly increasing. This implies that h'' is positive, and thus, by the second-order Taylor approximation, $h(a + x) > h'(a)x$.

We may now state Proposition 1:

Proposition 1. *Given constants n, X, x_i^-, x_i^+ with $1 \leq i \leq n$, such that $\sum_{i=1}^n x_i^+ \leq X \leq \sum_{i=1}^n x_i^-$, and n strictly convex, differentiable functions $f_i : R \to R$, there exists a unique solution to the optimization problem*

$$\min f(x_1, x_2, ...x_n) = \sum_{i=1}^{n} f_i(x_i)$$

subject to

$$\begin{aligned} \sum_{i=1}^{n} x_i &= X \\ x_i^- &\leq x_i \leq x_i^+ \end{aligned}$$

Furthermore this solution is exactly the point $x = (x_1, x_2, ..., x_n)$ for which there exists a λ such that

$$\begin{aligned} f_i'(x_i) &= \lambda \text{ when } x_i^- < x_i < x_i^+ &(1)\\ f_i'(x_i) &\leq \lambda \text{ when } x_i = x_i^+ &(2)\\ f_i'(x_i) &\geq \lambda \text{ when } x_i = x_i^- &(3) \end{aligned}$$

This solution can be directly located by varying X and applying the following rule: given X, if the associated solution x has $x_i = x_i^+$, then, for any $\widetilde{X} \geq X$, the associated solution $\widetilde{x}$ has $\widetilde{x}_i = x_i^+$ (the same applies to the x_i^-).

Proof: We first show that the characterization of minima given above is necessary. Given a point $x = (x_1, x_2, ..., x_n)$, will refer to a vector v as being admissible if there exists a positive ε^+ such that the line segment $l = \{x + \varepsilon v : \varepsilon \in [0, \varepsilon^+]\}$ lies within the domain. Let x be a minimum of f. Then, for any admissible v, the restriction of f to l, that is, the function $r(\varepsilon) = f(x + \varepsilon v)$, must have non-negative derivative at $\varepsilon = 0$.

Let x be a minimum of f. Suppose there were two indices $i \neq j$ for which $f_i'(x_i) > f_j'(x_j)$, $x_i^- < x_i < x_i^+$, and $x_j^- < x_j < x_j^+$; let $v = e_j - e_i$. Now $\left[\frac{d}{d\varepsilon} f(x + \varepsilon v)\right]_{\varepsilon=0} = f_j'(x_j) - f_i'(x_i) < 0$, a contradiction. Therefore we must have $f_i'(x_i) = f_j'(x_j)$ for interior coordinates.

Similarly, look at coordinates at their boundaries. Again, let x be a minimum of f. Suppose $x_i = x_i^+$. $v = e_j - e_i$ is admissible when $x_j < x_j^+$. Now $\left[\frac{d}{d\varepsilon} f(x + \varepsilon v)\right]_{\varepsilon=0} = f_j'(x_j) - f_i'(x_i) \geq 0$, and so $f_i'(x_i) \leq f_j'(x_j) = \lambda$. If $x_i = x_i^-$ then we have the similar result in (3) from looking at $v = e_i - e_j$.

Now we prove sufficiency. Suppose a point x satisfies the conditions (1), (2) and (3). Then consider any vector Δ such that $x + \Delta$ is within the domain. Note that $\sum_{i=1}^n (x_i + \Delta_i) = X$, and so $\sum_{i=1}^n (\Delta_i) = 0$. Now let U be the set of i for which $x_i = x_i^+$, let L be the set of i for which $x_i = x_i^-$ and let M be the set of i for which $x_i^- < x_i < x_i^+$. We compute

$$f(x + \Delta) - f(x) = \sum_{i=1}^{n} f_i(x_i + \Delta_i) - \sum_{i=1}^{n} f_i(x_i)$$

$$
\begin{aligned}
&= \sum_{i\in U} f_i(x_i+\Delta_i) + \sum_{i\in L} f_i(x_i+\Delta_i) \\
&\quad + \sum_{i\in M} f_i(x_i+\Delta_i) - \sum_{i=1}^{n} f_i(x_i) \\
&\geq \sum_{i\in U}(f_i(x_i) + f_i'(x_i)\Delta_i) + \sum_{i\in L}(f_i(x_i) + f_i'(x_i)\Delta_i) \\
&\quad + \sum_{i\in M}(f_i(x_i) + f_i'(x_i)\Delta_i) - \sum_{i=1}^{n} f_i(x_i) \\
&= \sum_{i\in U}(f_i'(x_i)\Delta_i) + \sum_{i\in L}(f_i'(x_i)\Delta_i) + \sum_{i\in M}(f_i'(x_i)\Delta_i)
\end{aligned}
$$

We now write, for $i \in U$, $f_i'(x_i) = \lambda - c_i$ with $c_i \geq 0$, and, for $i \in L$, $f_i'(x_i) = \lambda + c_i$ with $c_i \geq 0$. So

$$
\begin{aligned}
& \sum_{i\in U}(f_i'(x_i)\Delta_i) + \sum_{i\in L}(f_i'(x_i)\Delta_i) + \sum_{i\in M}(f_i'(x_i)\Delta_i) \\
&= \sum_{i\in U}((\lambda - c_i)\Delta_i) + \sum_{i\in L}((\lambda + c_i)\Delta_i) + \sum_{i\in M}(\lambda\Delta_i) \\
&= -\sum_{i\in U}(c_i\Delta_i) + \sum_{i\in L}(c_i\Delta_i) + \lambda\sum_{i\in M}\Delta_i \\
&\geq 0
\end{aligned}
$$

Hence the characterization above demonstrates necessary and sufficient conditions. Now, given a λ, we can pick a unique point x satisfying the characterization (1-3) as a direct result of the strict convexity of f_n, since f_n' is increasing. Since $\sum_{i=1}^{n} x_i = X$ is strictly increasing and continuous as a function of λ, we have an inverse on this domain. Thus for every X on this interval there is a unique λ and a unique point x where f attains its minimum, and this point meets the conditions (1-3).

Now consider X as an independent variable. Suppose that, for a certain value of X, the associated solution x has $x_i = x_i^+$. By the above claims, for a slightly larger X, the solution should not have its coordinate m decrease, and so it must remain constant. All other i such that $x_i < x_i^+$ should increase equally, by the previous section of the proof.

We now apply Proposition 2 to Problem 2. Given a total thermal load G, if the g_i^+ are sufficiently large, a minimum g will be interior, that is,

$$
f_i'(g_i) \quad = \quad 2\alpha_i g_i = \lambda, 1 \leq i \leq NT
$$

So $G = \sum g_j = \frac{\lambda}{2}\sum \alpha_j^{-1}$, and therefore g will have coordinates

$$
g_i = \frac{\lambda}{2\alpha_i} = \frac{G}{\alpha_i \sum \alpha_j^{-1}}
$$

Finally, computing cost at g, we have

$$c(g) = \sum \alpha_i g_i^2 = \frac{G^2}{\sum \alpha_i^{-1}}$$

Now consider G as an independent variable. Suppose that for some fixed m we have that $g_m^+ < g_i^+$ for all i. The solution above is a minimum for all $G < g_m^+ \alpha_m \sum \alpha_j^{-1}$; so, by continuity, it remains a minimum for $G = g_m^+ \alpha_m \sum \alpha_m^{-1}$. But in this case, $g_m = g_m^+$; that is, the minimum lies in the boundary of the domain. So, by Proposition 2, for values of G a bit greater than $g_m^+ \alpha_m \sum \alpha_m^{-1}$,

$$\begin{aligned} g_m &= g_m^+ \\ g_i &= \frac{G}{\alpha_i \sum_{j \neq m} \alpha_j^{-1}}, i \neq m \end{aligned}$$

And so the cost at g will be

$$c(g) = \alpha_m \left(g_m^+\right)^2 + \frac{G^2}{\sum_{i \neq m} \alpha_i^{-1}}$$

Summing up, we can consider $c(g)$ as a function p of the independent variable G. We have determined the expression for $p(G)$ above in two cases: first, when no coordinates are at their maximum, and second, when only one of them is. In general, p will be a piecewise quadratic function, whose curvature increases with each piece. This implies that it is a strictly convex function.

2.2 Hydro Plant Scheduling

We can now move on to the hydro plant scheduling problem. Remembering that $G_t = d_t - h_t$, we can then write the volumes as $v_t = v_0 - \sum_{i=0}^{t-1} q_i$ and so rewrite the problem as that of obtaining the q_t which solve

Problem 3

$$\min \sum_{t=0}^{NS-1} p(d_t - \rho(v_0 - \sum_{i=0}^{t-1} q_i) q_t)$$

$$\begin{aligned} \text{subject to } \sum_{t=0}^{NS-1} q_t &\leq v_0 - v^- \\ 0 &\leq q_t \leq q^+ \end{aligned}$$

With only one further simplification, though admittedly a physically inaccurate one, we can again make application of Proposition 2 to arrive at a solution. If we simply assume that power generated is independent of reservoir volume, that is if $\rho = \gamma$ a constant, then we have

Problem 4

$$\min \sum_{t=a}^{b-1} p(d_t - \gamma q_t)$$

$$\begin{aligned} \text{subject to } \sum_{t=a}^{b-1} q_i &= v_a - v_b \\ 0 &\leq q_i \leq q_i^+ \end{aligned}$$

which satisfies the hypotheses of Proposition 2. Since p is strictly convex we have that there exists a unique optimum point. If $0 < q_i < q_i^+$ for this point then we have that $\frac{d}{dq_t}(p(d_t - \gamma q_t)) = p'(d_t - \gamma q_t) \cdot (-\gamma) = \lambda$ for all t. Since p' is invertible (it is a strictly increasing continuous function, for p is strictly convex) we get that

$$q_t = \left(d_t - p'^{-1}\left(\frac{-\lambda}{\gamma}\right)\right)\gamma^{-1} = \frac{d_t}{\gamma} - c$$

with c a constant. We subject this to the restriction $\sum_{t=a}^{b-1} q_i = v_a - v_b$, to get

$$\begin{aligned} \sum_{t=a}^{b-1} q_t &= \sum_{t=a}^{b-1}(\frac{d_t}{\gamma} - c) = \gamma^{-1}\sum_{t=a}^{b-1} d_t - (b-a)\cdot c = v_a - v_b \\ \text{and so } c &= \frac{(\gamma^{-1}\sum_{t=a}^{b-1} d_t + v_b - v_a)}{(b-a)}. \end{aligned}$$

Non-interior optima are more difficult to characterize, but will still follow the λ restrictions (1-3). Thus we can use λ as a parameter for finding solutions, searching for the λ prescribing a point where $\sum_{t=a}^{b-1} q_i = v_a - v_b$. This is, of course, very similar to what happens in the solution to Problem 2.

While Problem 4 is tractable, even a simple ρ function makes $\frac{\partial f}{\partial q_t}$ dependent upon all q_i, and so we weren't able to solve Problem 3 as formulated.

3 Volume choices in a finite set

To compute approximations of the solution to Problem 3, a reasonable approach is to restrict choices of the q_t to multiples of a fixed interval δv, and then attempt to locate minima in the resulting finite lattice. That is,

Problem 5

$$\min \sum_{t=0}^{NS-1} p(d_t - \rho(v_0 - \sum_{i=0}^{t-1} q_i)q_t)$$

$$\text{subjectto} \sum_{t=0}^{NS-1} q_t \leq v_0 - v^-$$
$$q_t \in \{0, \delta v, 2\delta v, ..., q^+\}$$

The algorithm we wrote is based on the idea that solving for loose final volume on the interval from a to b can be split into solving two problems, one from a to $a+1$ and one from $a+1$ to b. To state this simply, one first solves for the optimal schedule and associated cost for all starting volumes at time $a+1$. This behavior is optimal on this interval, given the starting volume, independent of the schedule on the interval a to $a+1$. Thus to solve the overall optimum, we simply look at our actual starting volume, v_a, and look at all allowed volumes at v_{a+1}. In assessing the cost of each option, we take the cost of this outflow at time a and add the previously calculated cost of optimal behavior from $a+1$ to b. We then take the outflow at a which minimizes total cost, adjoin it to the optimal schedule from $a+1$ to b, thus forming the overall cost minimizing schedule.

This method has the excellent property of reducing a finite search in 47 dimensions to one in only 46 dimensions followed by one in two dimensions. But from this point we can again split the 46 dimensional problem, solving separately from $a+1$ to $a+2$ and $a+2$ to b. This process cascades until we are left with 46 two dimensional problems, rather than one massive 47 dimensional problem. This improvement amounts to making a computationally intractable problem tractable.

The algorithm thus starts by calculating the cost of all possible choices for all possible volumes at time $t = N_S - 1$. It then selects the least expensive outflow for each volume and records the outflow choice and cost for this volume at this time. We can now go back to $t = N_S - 2$. For each allowed volume v_{N_S-2} we check all possible outflows γ. To calculate cost of each outflow choice candidate, we calculate the operating cost at $t = NS - 2$ with that outflow and then add the cost of ideal behavior at $t = NS - 1$ with volume $v_{NS-2} - \gamma$. Again we select the least expensive outflow choice γ and its cost and associate it with the given volume. The process continues backward to $t = 0$, where the desired volume is known. The pseudocode is given below.

```
// given: v0, d[t]
The algorithm thus starts by calculating the c for each w in
{vmin,vmin + dv, ... vmax}
  m <- 0
  for each gamma in {0, dv, ... qmax}
    if vmax >= w - gamma >= vmin
      if p(d[t] - rho(w)*gamma) < p(d[t] - rho(w)*m)
        m <- gamma
  q[w,NS] <- m
```

```
  C[w,NS] <- p(d[t] - rho(w)*m)

for each t from NS-1 to 1
  for each w in {vmin,vmin + dv, ... vmax}
    m <- 0
    for each gamma in {0, dv, ... qmax}
      if vmax >= w - gamma >= vmin
        if (p(d[t] - rho(w)*gamma) + C[w-gamma,t+1]
              < p(d[t] - rho(w)*m) + C[w-m,t+1])
          m <- gamma
    q[w,t] <- m
    C[w,t] <- p(d[t] - rho(w)*m) + C[w-m,t+1]

m <- 0
for each gamma in {0, dv, ... qmax}
  if vmax >= v0 - gamma >= vmin
    if (p(d[t] - rho(w)*gamma) + C[v0-gamma,1]
          < p(d[t] - rho(w)*m) + C[v0-m,1])
      m <- gamma
q[v0,0] <- m
C[v0,0] <- p(d[t],rho(w)*m) + C[v0-m,1]
```

This method is still far from efficient, as it requires $O(n^2)$ inner loops for n elements in $\{v^-, v^- + \delta v, v^- + 2\delta v, \dots v^+\}$. For $n = 1600$, the simple perl script takes 12 minutes to execute on a 500 Mhz Pentium III processor running Linux. However, it is guaranteed to locate the minimum for the given choice of δv. Furthermore, increasing the number of volume points does little to alter the path in most cases, and barely improves optimum cost. Therefore, while we have not yet devised a bound on the error in this system, we suspect it to be negligible.

4 The continuous model

We now deal with the possibility of controlling plant operation at any point in time in an interval $[a, b]$, rather than only at a finite set of time points. This means our domain D will be a subset of the space of continuous functions from $[a, b]$ to R, instead of a subset of R^n. A function from D to R will sometimes be referred to as a functional.

So, based on Problem 3, we desire to find the point (now a function $q(t)$) that minimizes the cost functional F between times 0 and NS:

Problem 6

$$\min F(q) = \int_0^{NS} p(d(t) - \rho(v(0) - \int_0^t q(\tau)d\tau)q(t))dt$$

$$\text{subject to} \int_0^{NS} q(t)dt \leq v(0) - v^-, \quad 0 \leq q(t) \leq q^+$$

To solve this, we also reformulate Problem 3 as that of finding the minimum in a sub-interval (a, b) where $v(a)$ and $v(b)$ are fixed values. This restricts us to the domain $D = \left\{q \in C^0[a,b] : \int_a^b q(t)dt = v(a) - v(b), 0 \leq q(t) \leq q^+\right\}$. That is, we have

Problem 7

$$\min F(q) = \int_a^b p(d(t) - \rho(v(a) - \int_a^t q(\tau)d\tau)q(t))dt$$

$$\text{subject to} \int_a^b q(t)dt = v(a) - v(b), \quad 0 \leq q(t) \leq q^+$$

(For brevity, we shall write $v(t) = v(a) - \int_a^t q(\tau)d\tau$ and $k(t) = d(t) - \rho(v(t))$, so that the cost functional may be written as simply $\int_a^b p(k(t))dt$.)

Let us look at Problem 7. We will first restrict ourselves to the case of interior minima of F, that is, minima which satisfy the conditions on $q(t)$ strictly. We do this because, as we shall see later, a solution of Problem 7 in a certain interval must be a juxtaposition of interior minima in sub-intervals and horizontal line segments.

We will now obtain an explicit necessary condition for interior minima. Given a point q, we will refer to φ as being admissible if there exists a positive ε^+ such that the line segment $l = \{q + \varepsilon\varphi : \varepsilon \in [0, \varepsilon^+]\}$ lies within D. If q is a minimum of F, then, for any admissible φ, the restricted function from $[0, \varepsilon^+]$ to R, $r(\varepsilon) = F(q + \varepsilon\varphi)$, must have non-negative derivative at $\varepsilon = 0$. In particular, if q is interior, $r'(0) = 0$.

So let q be an interior minimum of F. Consider an admissible φ at q; it must satisfy $\int_a^b (q + \varepsilon\varphi)(t)dt = v(a) - v(b) = \int_a^b q(t)dt$, and therefore $\int_a^b \varphi(t)dt = 0$. So, following the remark above: $r(\varepsilon)$, the restriction of F to the direction of φ at q, must have derivative zero at $\varepsilon = 0$. This derivative is

$$\left[\frac{d}{d\varepsilon}F(q + \varepsilon\varphi)\right]_{\varepsilon=0}$$
$$= \int_a^b p'(k(t)) \left(\rho'(v(t))q(t)\varphi(t) + \rho(v(t)) \int_a^t \varphi(\tau)d\tau\right) dt$$

which, by straightforward integration by parts, yields

$$\begin{aligned}
&\int_a^b \left(p'(k(t))\rho(v(t)) + \int_a^t p'(k(\tau))\rho'(v(\tau))q(\tau)d\tau\right)\varphi(t)dt \\
= &\int_a^b f(t)\varphi(t)dt
\end{aligned}$$

where

$$f(t) = p'(k(t))\rho(v(t)) - \int_a^t p'(k(\tau))(\rho(v(\tau)))'d\tau$$

We have, then, that $\int_a^b f(t)\varphi(t)dt = 0$ for all admissible φ. Now, since, for any ψ, the function $\psi - \frac{1}{b-a}\int_a^b \psi(t)dt$ has integral zero and is therefore admissible, we have $\int_a^b f(t)\psi(t)dt = \frac{1}{b-a}\left(\int_a^b f(t)dt\right)\left(\int_a^b \psi(t)dt\right)$for all ψ. In particular, for arbitrary $c \in (a,b)$, taking

$$\begin{aligned}
\psi(t) &= \frac{1}{\varepsilon} \text{ for } t \in (c, c+\varepsilon) \\
&\quad 0 \text{ elsewhere}
\end{aligned}$$

we have $\int_a^b f(t)\psi(t)dt = \frac{1}{\varepsilon}\int_c^{c+\varepsilon} f(t)dt = \frac{1}{b-a}\left(\int_a^b f(t)dt\right)$. And so, as $\varepsilon \to 0$, $f(c) = \frac{1}{b-a}\left(\int_a^b f(t)dt\right)$ for arbitrary c. Thus f is actually a constant function, that is,

$$f(t) = f(a)$$

meaning $p'(k(t))\rho(v(t)) - \int_a^t p'(k(\tau))(\rho(v(\tau)))'d\tau = p'(k(a))\rho(v(a))$. So, differentiating, $(p'(k(t)))'\rho(v(t)) = 0$. This implies that $p' \circ k$ is constant, and so $p'(k(t)) = p'(k(a))$. Since p is strictly convex, this means

$$k(t) = k(a)$$

That is, $d(t) - \rho(v(t))q(t) = d(a) - \rho(v(a))q(a)$. This may be formulated as the ODE/initial value problem

$$\begin{aligned}
v'(t) &= \frac{d(a) - d(t) - \rho(v(a))q(a)}{\rho(v(t))} \\
v(a) &= c
\end{aligned}$$

This means, given $q(a)$, there is a certain $v(b)$ for which the ODE above characterizes the solution. (This induced the numerical method described below.) By the existence and uniqueness theorem for ODEs, this is a complete characterization of the solution. Under a certain hypothesis, the uniqueness of a solution to Problem 7 is guaranteed, and being a solution to the initial value problem above becomes a sufficient condition. We give two example cases where the problem simplifies somewhat and we have analytic solutions.

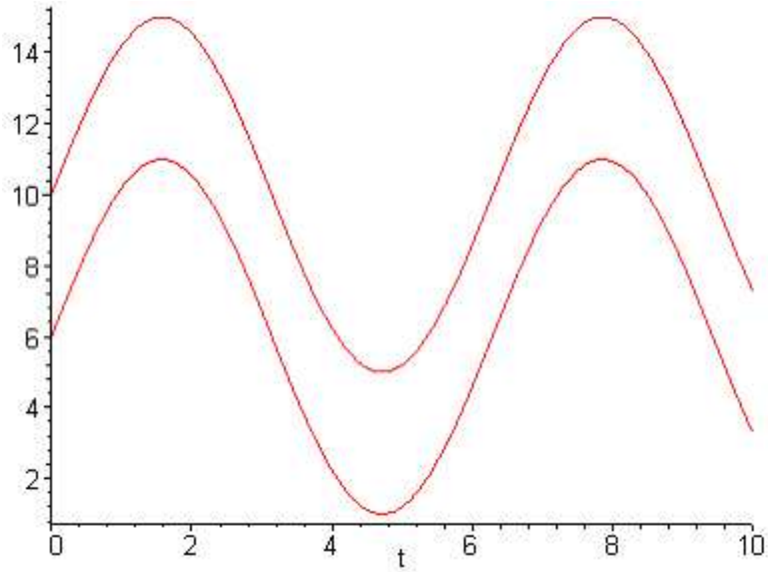

Figure 1: Cost minimizing $q(t)$ for $d(t) = 10 + 5\sin t$

4.0.1 Example 1

If $\rho \equiv 1$ (admittedly a very unreasonable assumption), then the problem reduces drastically, namely to that of solving $v' = d(a) - q(a) - d(t), v(b) = c$. (Assume that $-v' < q^+$.) This yields $v(t) = \int_a^t d(\tau)d\tau + (d(a) - q(a))\,(t - a) + \left[c - \int_a^b d(\tau)d\tau + (d(a) - q(a))\,(b-a)\right]$ Figures 1 and 2 ($d(t)$ and $q(t)$ on 1, $v(t)$ on 2) are for the following data: $d(t) = 10 + 5\sin t, a = 0, b = 10, q(a) = 6, v(a) = 70$.

In this case, in fact, we may prove directly that a solution of the ODE is a global minimum: here f becomes a mere $p'(d(t) - q(t))$, and, given any $\tilde{q}$ in the domain, we have

$$
\begin{aligned}
& \int_a^b (p(d(t) - \tilde{q}(t)) - p(d(t) - q(t)))\,dt \\
\geq\; & \int_a^b p'(d(t) - q(t))\,((d(t) - \tilde{q}(t)) - (d(t) - q(t)))\,dt \\
=\; & \int_a^b p'(d(t) - q(t))(q(t) - \tilde{q}(t))dt \\
=\; & -\int_a^b f(t)(\tilde{q} - q)(t)dt
\end{aligned}
$$

But of course, $\tilde{q} - q$ is an admissible variation, and so $\int_a^b f(t)(\tilde{q} - q)(t)dt = 0$. Thus the cost of any $\tilde{q}$ is greater than the cost of q.

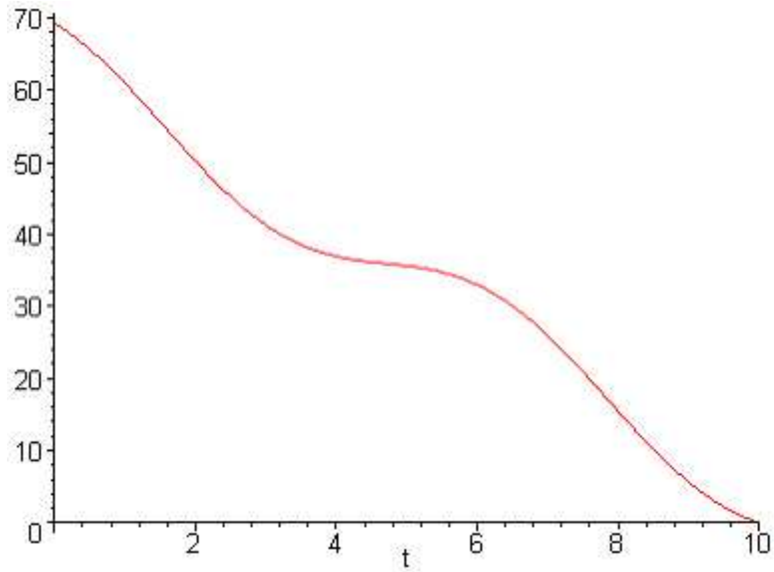

Figure 2: Cost minimizing $v(t)$ for $d(t) = 10 + 5\sin t$

4.0.2 Example 2

If $\rho(t) = \sqrt{x}$ and, additionally, $d \equiv D$ (a constant) and b is such that $\frac{\sqrt{v(a)}q(a)}{\sqrt{v(b)}} < q^+$, we have $v' = -\frac{\sqrt{v(a)}q(a)}{\sqrt{v}}, v(b) = c$. The solution is then

$$v(t) = \left(2v(a)^{3/2} - 3\sqrt{v(a)}q(a)(t-a)\right)^{2/3}.$$

Figures 3 and 4 are for arbitrary d, $v(a) = 1, q(a) = 1, a = 0, b = 0.9, q^+ \geq 4.65$. (Note that, for this formula, $q \to \infty$ as b approaches 1. This is an example where a portion of the optimum path is interior, but it will necessarily hit the q^+ boundary for instants close to 1.)

4.1 Boundary minima

Now that we have characterized interior solutions of Problem 7, we consider the possibility that a solution q in $[a, b]$ will be such that $q(t) = q^+$ or $q(t) = 0$ for certain intervals of t. Certainly, if we restrict q to a sub-interval $[a', b']$ in which $q^+ > q(t) > 0$, it should satisfy the corresponding ODE for this interval. That is, q should consist of a succession of curves given by the ODE and horizontal line segments with values 0 and q^+.

Indeed, we can easily give a restricted analogue to Proposition 2, that is, a sufficient condition for a q with the property above to be a local minimum, that is, for all restricted functions in the direction of admissible directions to have

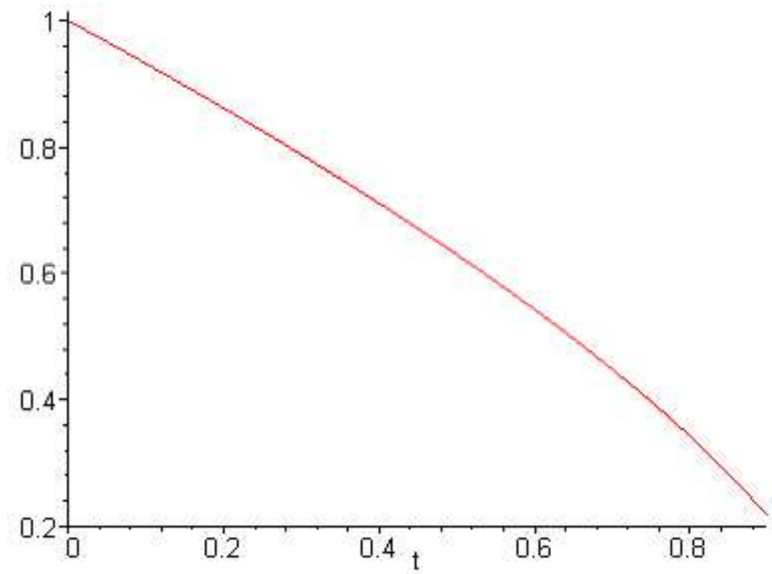

Figure 3: Cost minimizing $v(t)$ for $v(a) = 1, q(a) = 1, a = 0, b = 0.9, q^+ \geq 4.65$.

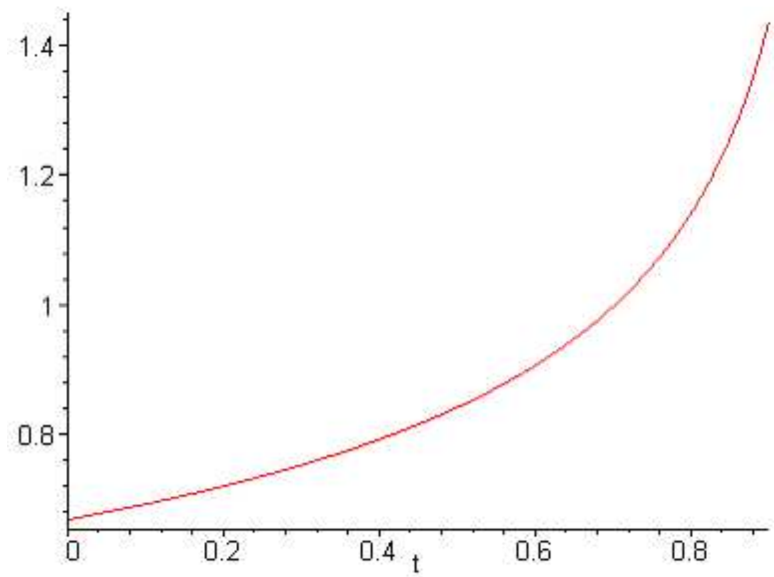

Figure 4: Cost minimizing $q(t)$ for $v(a) = 1, q(a) = 1, a = 0, b = 0.9, q^+ \geq 4.65$.

positive derivatives at zero. Let $L = q^{-1}(0)$, $U = q^{-1}(q^+)$, $M = [a,b] - (L \cup U)$. (As stated above, each of these is a finite union of intervals.) Note that now, for a variation φ to be admissible, it must not only satisfy $\int_a^b \varphi(t)dt = 0$, but also $\varphi(L) \geq 0$ and $\varphi(M) \leq 0$. For such φ, assume that $\left[\frac{d}{d\varepsilon}F(q+\varepsilon\varphi)\right]_{\varepsilon=0}$ can be written as $\int_a^b f(t)\varphi(t)dt$, as before.

So, if there is a constant λ such that

$$\begin{aligned} f(M) &= \lambda, \\ f(L) &\geq \lambda \text{ and} \\ f(U) &\leq \lambda, \end{aligned}$$

then we can write $f = \lambda + r$, with $r(L) \geq 0$ and $r(U) \leq 0$, and we have

$$\begin{aligned} \int_a^b f(t)\varphi(t)dt &= \int_a^b \lambda\varphi(t)dt + \int_L r(t)\varphi(t)dt + \int_U r(t)\varphi(t)dt \\ &\geq 0, \end{aligned}$$

proving that the requirement above is indeed a sufficient condition for local minima. (For the $\rho \equiv 1$ case, these points are actually global minima also, for the same argument used above will still apply here.)

4.2 Extensions and numerical results

Ideally, we would like to apply the above developments to a complete solution of Problem 6. However, the best we could do was the following: if the machinery above gives us, for each $v = v(b)$, a function q_v which is indeed the solution of Problem 7 in $[0, NS]$ for this $v(b)$, then what we have left is an ordinary one-variable calculus problem:

$$\begin{aligned} \min F(v) &= \int_0^{NS} p(d(t) - \rho(v(a) - \int_0^t q_v(\tau)d\tau)q_v(t))dt \\ \text{subject to } v^- &\leq v \leq v^+ \end{aligned}$$

We thus wrote the following algorithm. For arbitrary values of $q(a)$, it attempts to follow the ODE (with Euler's method) where it prescribes admissible q values, and sticks to the boundary otherwise. It then either reaches $v = 0$ at some $t_0 < b$ (in which case the process is repeated for a smaller value of $q(a)$, for this t_0 can be interpreted as a strictly decreasing function of $q(a)$), or reaches some $v = v(b) \geq 0$. We then compute the total cost of the corresponding q_v, and attempt to "home in" on the minimum total cost. (This assumes that the cost of the optimum path starting at $v(a), q(a)$ is a well-behaved function of $q(a)$.)

Figures 5 and 6 show numerical results (in black) against theoretical data (in red) for the data $a = 1, b = 48, v(a) = 100, q(a) = 1.449$. $q(a)$ was obtained numerically for minimum cost. The algorithm's pseudocode follows below.

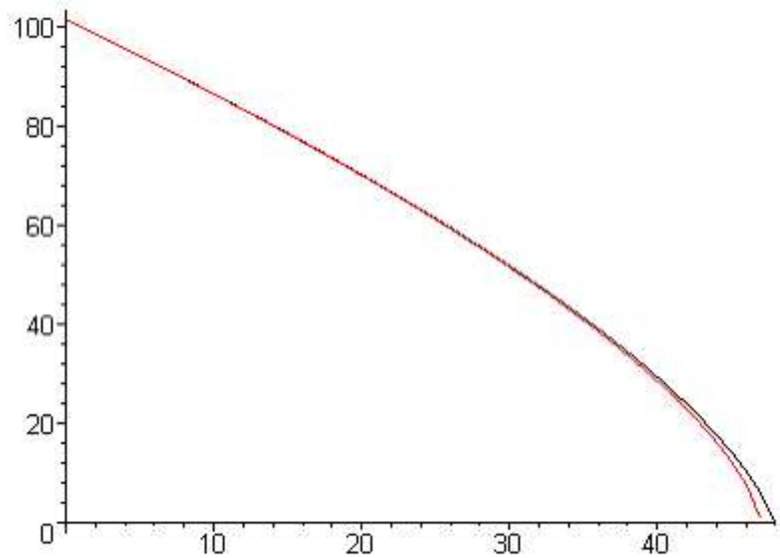

Figure 5: Numerical vs theoretical v(t) for $a = 1, b = 48, v(a) = 100, q(a) = 1.449$

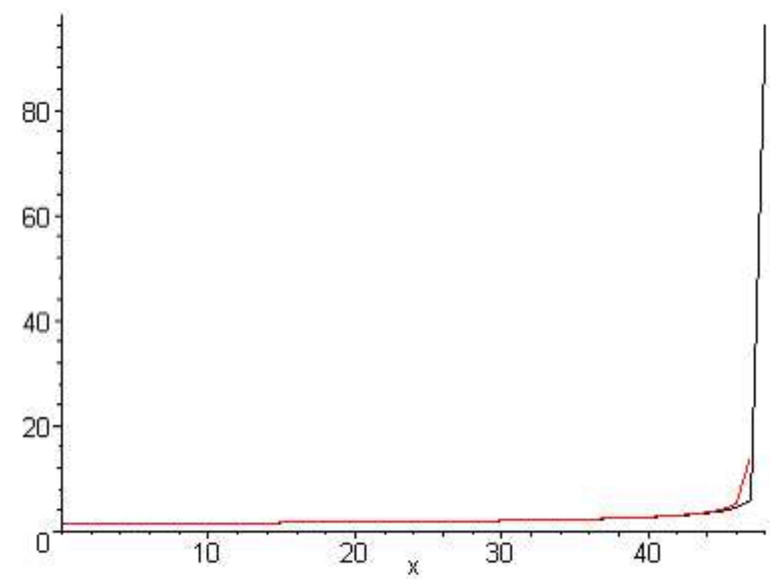

Figure 6: Numerical vs theoretical q(t) for $a = 1, b = 48, v(a) = 100, q(a) = 1.449$

```
qupper <- qmax; qlower <- 0
for each dq in {1,(1/10),...(1/10^p)
  for each q[0] in {qlower, qlower + dq, ... qupper}
    for each t from 1 to NS do
      qcandidate <- (rho(v[0]))*q[0] + d[t] - d[0])/rho(v[t])
      if qcandidate > qmax
        qcandidate <- qmax
      if qcandidate < 0
        qcandidate <- 0
      if v[t] + qcandidate > 0
        v[t+1] <- v[t] + qcandidate
      else this solution spends too much and we discard it
    compute cost[q[0]]
  find q0min in {qlower, qlower + dq, ... qupper} that minimizes cost[q0]
  qupper <- q0min + 2*dq
  qlower <- q0min - 2*dq
```

5 Acknowledgements

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