An Analysis of the Collatz Conjecture

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1 Introduction

The Collatz conjecture remains today unsolved; as it has been for over 60 years. Although the problem on which the conjecture is built is remarkably simple to explain and understand, the nature of the conjecture and the behavior of this dynamical system makes proving or disproving the conjecture exceedingly difficult. As many authors have previously stated, the prolific Paul Erdos once said, "Mathematics is not ready for such problems." Thus far all evidence indicates he was correct (see [2]). Although the conjecture may be stated in a variety of ways this paper will focus on the modified Collatz map $T : \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by

$$T(n) = \begin{cases} \frac{3n+1}{2} & n \equiv 1 \pmod 2 \\ \frac{n}{2} & n \equiv 0 \pmod 2 \end{cases}$$

The conjecture is that for every integer n there exists a k such that $T^k(n)$=1.

First we will take steps toward refining an upper bound of the growth of a divergent trajectory if one exists. We will also look at an interesting pattern found when counting the number of iterations of certain operations on numbers of the form $(2^p+1)^m-1$, for some $p > 1$. Lastly we will consider a continuous extension of this map to the real line.

Although this paper is not exactly a bibliography of previous works we will reference known results. This survey will document the course we took to analyze the Collatz problem and provide our own proofs even for known results.

2 Divergent Trajectories

Following Lagarias' suggestion (see [5]) we sought an upper bound for the growth of a divergent trajectory, if one exists. The most obvious upper bound comes from the observation that the fastest possible growth occurs in a monotonically increasing trajectory of odd numbers.

By iterating $\alpha(n) = \frac{3n+1}{2}$ we find the following upper bound

$$\alpha^k(n) = \left(\frac{3}{2}\right)^k n + \sum_{i=0}^{k-1} \frac{3^i 2^{k-i-1}}{2^k},$$

which simplifies to the geometric series:

$$\left(\frac{3}{2}\right)^k n + \frac{1}{2}\sum_{i=0}^{k-1} \left(\frac{3}{2}\right)^i.$$

Upon further simplification we find

$$\alpha^k(n) = \left(\frac{3}{2}\right)^k (n+1) - 1. \tag{1}$$

Let $\phi(n)$ be the first k such that $\alpha^k(n)$ is an even integer. If a monotonically increasing trajectory exists starting with n_0 then there is no largest k for n_0 and $\phi(n_0)$ is undefined. From equation (1) no such trajectory exists. The exact behavior of $\phi(n)$ we describe in Theorem 1.

Theorem 1. *If p_0 is such that*

$$p_0 \equiv -1 \ (mod\ 2^n),$$

where n is the largest integer such that this congruence holds, then

$$\phi(p_0) = n.$$

Proof. Although this theorem follows from (1) by observing that n+1 $\equiv$ 0 $(mod\ 2^k)$, k-maximal implies that $\alpha^k(n)$ is even and for $1 \leqslant i < k$, $\alpha^i(n)$ is odd, we offer an alternative proof:

Suppose without loss of generality that p_0 is odd. Then

$$3p_0 + 1 \equiv -2 \ (mod\ 2^n),$$

hence

$$p_1 \equiv \frac{3p_0+1}{2} \equiv -1 \ (mod\ 2^{n-1}).$$

Repeating this argument we get that

$$p_2 \equiv -1 \ (mod\ 2^{n-2})$$

$$\vdots$$

$$p_{n-1} \equiv -1 \ (mod\ 2).$$

Since all those p_j are odd, this gives us that $\phi(p_0) \geqslant n$.

Suppose now that $\phi(p_0) > n$, i.e.

$$p_n \equiv -1 \ (mod\ 2).$$

But $2p_n - 1 = 3p_{n-1}$, and since $2p_n \equiv -2 \ (mod\ 2^2)$

$$p_{n-1} = \frac{2p_n - 1}{3} \equiv -1 \ (mod\ 2^2).$$

Repeating this reasoning we get that

$$p_0 \equiv -1 \ (mod\ 2^{n+1}),$$

which contradicts the fact that n is maximal.

$\square$

Corolary $\phi(m)$ is finite for every m.

Corolary For every natural number k there are infinitely many numbers n such that $\phi(n) = k$.

Proof. Just take $n = l \cdot 2^k - 1$, where l is an odd number. $\square$

From the formula (1) it is easy to see that $\alpha^k(n) \equiv 2 \ (mod\ 3)$ for all k such that this number is an integer. In particular $\alpha^{\phi(n)}(n) \equiv 2 \ (mod\ 3)$, but since (from the definition of $\phi(n)$) it is also even, we have that

$$\alpha^{\phi(n)}(n) \equiv 2 \ (mod\ 6).$$

But now, since every even number is eventually taken to an odd number by successive applications of the function T, and T executes the operation α on an odd number m exactly $\phi(m)$ times, we deduce that every number is taken to a number congruent to 2 $(mod\ 6)$, which gives us the following theorem:

Theorem 2. *In order to prove the Collatz conjecture, it is sufficient to prove it for every number congruent to* 2 $(mod\ 6)$*.*

Since $\phi(m)$ is finite for every integer m it is not possible for an unbounded trajectory to consist entirely of odd numbers and thus our initial upper bound can be improved. From Theorem 1 we conclude that if $m \leqslant m_0 = 2^k - 1$ for some integer k, then $\phi(m) \leqslant k = \log_2(m_0 + 1)$. After $\phi(m)$ applications of α we must divide the result by 2 at least once. Since our goal is an upper bound, we will assume division by 2 occurs exactly once and that this process continues indefinitely. In other words let us assume that $m_k = \frac{\alpha^{\phi(m_{k-1})}(m_{k-1})}{2}$ is odd for all $k \geqslant 1$. We know $\phi(m_0) \leqslant \log_2(m_0 + 1)$, so when we iterate this process:

$$m_0 = m$$
$$m_1 \leqslant \frac{\left(\frac{3}{2}\right)^{\log_2(m_0)}(m_0+1)-1}{2}$$
$$\vdots$$
$$m_n \leqslant \frac{\left(\frac{3}{2}\right)^{\log_2(m_{n-1})}(m_{n-1}+1)-1}{2}$$

Already m_2 is quite complicated. We were unable to find a closed form equation for this recursively defined upper bound. Instead we choose a particular starting integer m and apply the following algorithm to demonstrate the growth according to this new upper bound:

Given a starting integer m_0 recursively define $m_{n+1} = (3m_n + 1)/2$ for $0 \leqslant n < \text{ceiling}(\log_2(m_0 + 1))$. Then define $m_{n+2} = \frac{m_{n+1}}{2}$. Repeat the process, now starting at m_{n+2}, applying $(3m_k + 1)/2$ exactly ceiling$(\log_2(m_{n+2} + 1))$-times.

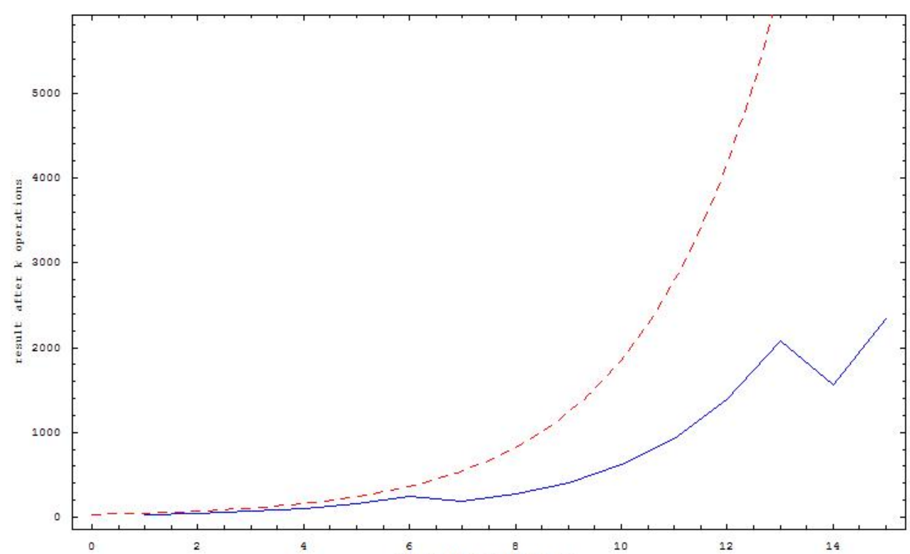

Figure 1:

Figure 1 shows a graph of the function $(3/2)^k(m_0+1)-1$ where $m_0 = 31$ (red-dashed) and a plot of points found by the above algorithm (blue-solid).

Since there are estimates on lower bounds for the growth of a divergent trajectory (see [5]), finding a sufficiently small upper bound would lead to a proof that no divergent trajectory exist. Also Theorem 2 and equation (1) will be useful in developing optimizations for algorithms which test the conjecture for all values less than some large integer. Although such computer programs will never result in a proof of the conjecture, they can be used to obtain minimum lengths of non-trivial cycles.

3 Further Investigations

The natural step after the investigation of the behavior of the function ϕ would be to investigate the function $\psi(m) : \mathbb{N} \to \mathbb{N}$, which we define as the highest power of 2 that divides m. If we consider the natural ordering of even numbers, $\psi(m)$ results in the same pattern as $\phi(n)$ acting on odd numbers in their natural ordering:

m	$\phi(m)$	n	$\psi(n)$
1	1	2	1
3	2	4	2
5	1	6	1
7	3	8	3
9	1	10	1
11	2	12	2
13	1	14	1
15	4	16	4
17	1	18	1
19	2	20	2
21	1	22	1

However if one starts with odds in their natural order and applies α until the first even integer appears, and then considers the ψ function a much more complicated pattern emerges. In fact it appears that no pattern exists at all so we tried to understand the pattern which arises when we restrict ourselves to certain subsets of the odd numbers. Our first investigation concerned numbers of the form $3^n - 1$, because these are the numbers obtained from $\phi(2^n - 1)$ iterations of T on $2^n - 1$. Theorem 1 gives us that numbers of the form $2^n - 1$ grow the most of any number less than or equal to $2^n - 1$

before shrinking the first time. We observed that a similar pattern to the abovemay exist when we restrict ourselves to this subset of odd numbers.

Then we investigated the pattern of $\psi(p^m - 1)$ where $p = 2^n + 1$. This is the subject of this section.

m	$\phi(2m-1)$	$\psi(5^m-1)$	$\psi(9^m-1)$
1	1	2	3
2	2	3	4
3	1	2	3
4	3	4	5
5	1	2	3
6	2	3	4
7	1	2	3
8	4	5	6
9	1	2	3
10	2	3	4
11	1	2	3
12	3	4	5
13	1	2	3
14	2	3	4
15	1	2	3
16	5	6	7
17	1	2	3
18	2	3	4
19	1	2	3
20	3	4	5
21	1	2	3
22	2	3	4
23	1	2	3
24	4	5	6
25	1	2	3
26	2	3	4
27	1	2	3
28	3	4	5
29	1	2	3
30	2	3	4
31	1	2	3
32	6	7	8

m	$\phi(2m-1)$	$\psi(5^m-1)$	$\psi(9^m-1))$
33	1	2	3
34	2	3	4
35	1	2	3
36	3	4	5
37	1	2	3
38	2	3	4
39	1	2	3
40	4	5	6
41	1	2	3
42	2	3	4
43	1	2	3
44	3	4	5
45	1	2	3
46	2	3	4
47	1	2	3
48	5	6	7
49	1	2	3
50	2	3	4
51	1	2	3
52	3	4	5
53	1	2	3
54	2	3	4
55	1	2	3
56	4	5	6
57	1	2	3
58	2	3	4
59	1	2	3
60	3	4	5
61	1	2	3
62	2	3	4
63	1	2	3
64	7	8	9

Lemma 1. *Let $m \in \mathbb{Z}$ and $n > 1$. Let k be the greatest number that satisfies $m \equiv 0(mod\ 2^k)$. Then* $\left(\begin{array}{c} m \\ i \end{array} \right) 2^{in} \equiv 0(mod\ 2^{k+n+1})$, *for* $1 < i \leqslant m$.

Proof. This is the same as proving that

$$\alpha = \frac{m}{2^k} \left(\begin{array}{c} = m-1 \\ i-1 \end{array} \right) \frac{2^{(i-1)n}}{i}$$

is even. But $m/2^k$ is an integer, so is $\left(\begin{array}{c} m-1 \\ i-1 \end{array} \right)$. Then the only problem is $2^{(i-1)n}/i$. But since $i \leqslant 2^{(i-1)}$, if $i > 1$ the numerator of the fraction must contain at least one more factor of 2 then the denominator. Since

$$\left(\begin{array}{c} m \\ i \end{array} \right) 2^{in}$$

is an integer, the denominator of α must be a power of 2. Since we proved that there are more factors of 2 in the numerator than in the denominator, this proves the lemma. $\square$

Theorem 3. *Let $p = 2^n + 1$, for $n > 1$. Let $\psi(p^m - 1)$ be the number of times that $p^m - 1$ is divisible by 2. Let k be the greatest number that satisfies $m \equiv 0($ mod $2^k)$. So*

$$\gamma_n(m) = n + k, \quad \textit{for} \quad m \geqslant 1.$$

Proof. In order to prove the theorem, we separate two cases:

First Case: m odd.

We have that

$$\begin{aligned} p^m - 1 &= (p-1)(p^{m-1} + p^{m-2} + ... + p + 1) \\ &= 2^n(p^{m-1} + p^{m-2} + ... + p + 1). \end{aligned}$$

Since p is odd, we have that p^i is odd for every i. Hence

$$(p^{m-1} + p^{m-2} + ... + p + 1)$$

is odd. Therefore,

$$p^m - 1 \equiv 0\ (mod\ 2^n),$$

where n is maximal. Since m is odd, we have that $m \equiv 0(\text{mod}2^k)$ if and only if $k = 0$. Hence $\psi(p^m - 1) = n$.

Second Case: m even.
We know, from our hypothesis, that

$$m2^n \equiv 0 \pmod{2^{n+k}}$$

$n+k$ is maximal with that property. In other words,

$$m2^n = 2^{n+k}z, \text{with z odd.}$$

Now, from lemma 1:

$$\binom{m}{i} 2^{in} = 2^{n+k}b_i, \text{ com } b_i \text{ par } (1 < i \leqslant m).$$

Hence $z + b_2 + ... + b_m$ is an odd number. And since

$$p^m - 1 = 2^{n+k}(z + b_2 + ... + b_m),$$

we have that:

$$p^m - 1 \equiv 0(\mod 2^{k+n})$$

and $n+k$ is the largest possible that satisfies this equation. Therefore $\psi(p^m - 1) = k + n$, as we wished. □

4 Mixing Properties of the T map

If we choose an odd integer m_0 at random and iterate T until the next odd number appears, it is not difficult to see that the next odd will occur after one iteration with a probability of 1/2, in other words $m_1 = (3m_0 + 1)/2$ 1/2 of the time. Similarly the probability that $m_1 = (3m_0 + 1)/4$ is 1/4 and in general $m_1 = (3m_0 + 1)/(2^n)$ with probability $1/(2^n)$. Now if we assume that T acts in such a way that successive odds appear as if drawn at random then one can find the expected growth between successive odds.

$$\left(\frac{3}{2}\right)^{\frac{1}{2}} \left(\frac{3}{4}\right)^{\frac{1}{4}} \left(\frac{3}{8}\right)^{\frac{1}{8}} ... = \frac{3}{4}.$$

This is the expected growth of an odd integer to the next odd, and so we expect the sequence to shrink. This argument supports the Collatz conjecture but relies on the assumption that successive odds appear at random under the action of T. The following section discusses one known result which seems to support this conjecture.

Definition We will let $x_i(m)$ be the binary sequence defined by

$$T^i(m) \equiv x_i(m) \ (mod\ 2).$$

Theorem 4. *Let*

$$v_k(m) = (x_0(m), x_1(m), \dots x_{k-1}(m)).$$

Now fix k. If we make m run over any set of 2^k consecutive numbers, $v_k(m)$ will run over every possible binary combination exactly once.

Proof. We will reason by induction on k. The theorem is obvious for $k = 1$. Now suppose that it is true for some fixed k. Notice that this implies that $v_k(m) = v_k(m+2^k)$, because if we let n run over the interval $m \leqslant n < m+2^k$, $v_k(n)$ will run over all possible binary k-tuples exactly once, and the same will happen if we let n run over the interval $m < n \leqslant m + 2^k$. But the intersection of those two sets contains $2^k - 1$ numbers, which leaves only one possible value to be assumed by both $v_k(m)$ and $v_k(m + 2^k)$.

Now take two numbers m and $m + l \cdot 2^j$ where l is an odd number, and $j \geqslant 1$. Then if m is even so is $m + l \cdot 2^j$ and

$$T(m) = \frac{m}{2}, \qquad T(m + l \cdot 2^j) = \frac{m}{2} + l \cdot 2^{j-1} = T(m) + l \cdot 2^{j-1},$$

and if m is odd,

$$T(m) = \frac{3m+1}{2}, \qquad T(m + l \cdot 2^j) = \frac{3m+1}{2} + 3l \cdot 2^{j-1} = T(m) + k \cdot 2^{j-1},$$

where k is again odd. Now observe that $x_j(m)$ defines which operation will be executed by T on $T^j(m)$. Therefore, since $v_k(m) = v_k(m + 2^k)$, T will execute the same operation over both numbers k times, and from the observation above, we will get

$$T^k(m) = T^k(m + 2^k) + l,$$

where l is an odd number. Therefore

$$T^k(m) \not\equiv T^k(m + 2^k) \ (mod\ 2).$$

This shows that if we take a set of 2^{k+1} consecutive numbers and fix a binary k-tuple $v = (v_0, v_1, \dots v_{k-1})$, since v is the image under v_k of two numbers m and $m+2^k$ in this set, we can find the $k+1$-tuples $(v_0, v_1, \dots v_{k-1}, 0)$ and $(v_0, v_1, \dots v_{k-1}, 1)$ in the image of this set under v_{k+1}, and that proves the theorem. $\square$

Although the above result is interesting it is not enough to prove the conjecture. This theorem merely shows that no two starting values from any 2^k consecutive integers share the same sequence of the first k-operations by T.

5 Dynamical System approach

As an exercise we sought to create a continuous extension of the Collatz map to the complex plane and follow the work of Chamberland [1]. In that article Chamberland extends the Collatz map to a function $f : \mathbb{R} \to \mathbb{R}$ defined to be:

$$f(x) = \frac{3x+1}{2} sin^2\left(\frac{\pi x}{2}\right) + \frac{x}{2} cos^2\left(\frac{\pi x}{2}\right).$$

We pursued a function of the form:

$$g(x) = \frac{p(r(x))}{q(r(x))} x + r(x),$$

Which led to $g : \mathbb{C} \to \mathbb{C}$

$$g(x) = \frac{2 + (-1)^{x+1}}{2} x + \frac{(-1)^{x+1} + 1}{4}$$

What we found is that if we restrict this function to the real axis, $g(x)$ is identically $f(x)$.

Proposition 1. *$Re(g(x)) = f(x)$.*

Proof. First observe:

$$g(x) = \left(x + \frac{1}{4}\right) - \left(\frac{2x+1}{4}\right) cos(\pi x) - \left(\frac{2x+1}{4}\right) sin(\pi x) i$$

and therefore

$$Re(g(x)) = \left(x + \frac{1}{4}\right) - \left(\frac{2x+1}{4}\right) cos(\pi x)$$

But

$$\begin{aligned}
f(x) &= -\left(x+\frac{1}{2}\right)cos^2\left(\frac{\pi x}{2}\right)+\frac{3x+1}{2} \\
&= -\left(x+\frac{1}{2}\right)\left(\frac{1+cos(\pi x)}{2}\right)+\frac{3x+1}{2} \\
&= -\left(\frac{2x+1}{4}\right)(1+cos(\pi x))+\frac{3x+1}{2} \\
&= \left(x+\frac{1}{4}\right)-\left(\frac{2x+1}{4}\right)cos(\pi x) = Re(g(x))
\end{aligned}$$

□

Using a bisection algorithm we numerically approximated solutions to $f(f(x)) - x = 0$. Clearly these values approximate two-cycles and one-cycles. With this list we can analyze the behavior of the two cycles by differentiating $f(f(x))$. If $|f(f(x_0))'| < 1$ we say $(x_0, f(x_0))$ is an attractor. We verified Chamberland's findings that c_1 :=(1,2), the trivial cycle and c_2 :=(1.192531907..., 2.138656335...) are attractors (see [1]). Chamberland conjectures that these are the only attracting two-cycles on $\mathbb{R}^+$. Although the complexity of $f(f(x))'$ seems to prohibit proving this conjecture by analyzing the derivative explicitly, we numerically verified that no other two-cycle with an element less than 1000 is an attractor.

In addition to the analysis of the real continuous extension we sought the set of points in the complex plane which converge. Picking an $\epsilon > 0$ we say a point $z \in \mathbb{C}$ has converged if after n iterations of f, $|f^n(z) - (c_i)_k| < \epsilon$. This set is a fractal domain of convergence in the complex plane as seen in Figures 2 and 3.

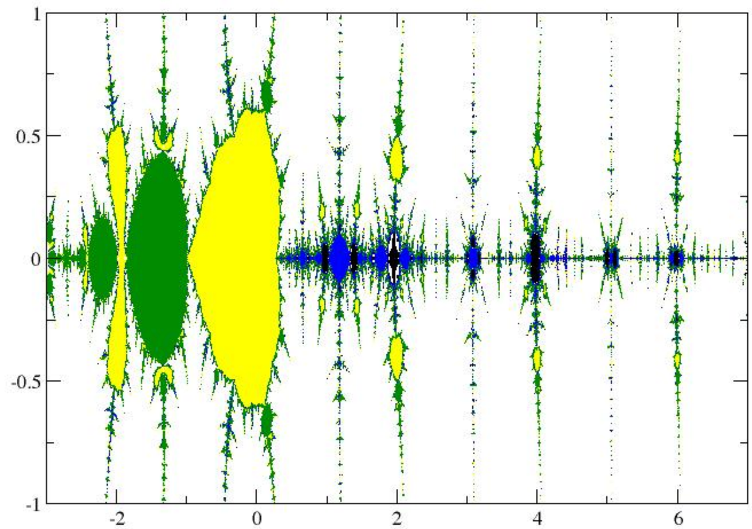

Figure 2:

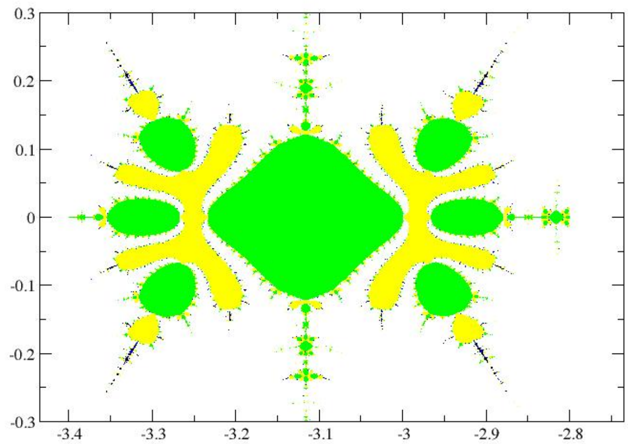

Figure 3:

Research on this problem is still very active. There are many different ways to approach this problem, a handful of which we have presented. We hope to continue investigating this problem and further develop the results discussed in this paper. For a more detailed bibliography see Lagarias' annotated bibliography [6].

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