Problem of the Week

Proposed by Bernardo Ábrego and Silvia Fernández. **May 2-9**

The following is a well known fact:

> If a positive integer is not prime then it has a prime factor less than or equal its square root.

Show that a positive integer having more than four positive divisors has a prime factor less than its cubic root. In other words, show that every positive integer n, with more than four positive divisors, has a prime factor p such that $p < \sqrt[3]{n}$.

Solution by the organizers. If n is a power of a prime, $n = p^a$, then the positive divisors of n are $1, p, p^2, \ldots, p^a$. By assumption n has more than four positive divisors so $a \geq 4$ and then $n = p^a \geq p^4$, thus $p \leq \sqrt[4]{n} < \sqrt[3]{n}$.

If n is not a power of a prime and has only two different prime factors then $n = p_1^a p_2^b$ with $p_1 < p_2$ prime numbers. If $a = b = 1$ then the only positive divisors of n are $1, p_1, p_2$, and $p_1 p_2$. By assumption n has more than four positive divisors, so at least one of a or b is at least 2. That is $a + b \geq 3$, and then $n = p_1^a p_2^b > p_1^{a+b} \geq p_1^3$, or $p_1 < \sqrt[3]{n}$.

Finally, if n has three or more distinct prime factors then $n = p_1 p_2 p_3 N$ with $p_1 < p_2 < p_3$ prime numbers. Then $n = p_1 p_2 p_3 N > p_1^3$ or $p_1 < \sqrt[3]{n}$.