California State University
**Northridge**

# Problem of the Week

Department of Mathematics

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Jack wants to write a sum of numbers equal to 100 by using each of the digits 1 to 9 exactly once. His best try so far is:

$$4 + 6 + 9 + 18 + 25 + 37 = 99.$$

Show that, no matter how hard he tries, he will never succeed.

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**Solution 1 by Paul Ryan.** First, notice that there can only be one-digit or two-digit numbers in any combination that is a solution, because numbers with 3 or more digits will cause the sum to be more than 100. Further, notice that there must be at least one two-digit number, because $1+2+3+4+5+6+7+8+9 = 45$. And notice that there can be at most three two-digit numbers, because $15+26+37+48+9 = 135$ is the smallest combination with four two-digit numbers.

Now let's find out what the solutions containing one, two, or three two-digit numbers look like (if any exist). In order to add one two-digit number we take the all one-digit combination: $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$, subtract out one of the single digit numbers, call it $a$, and add it back in the tens place of one of the other numbers. So for our solution with one two-digit number we get:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 - a + 10a = 45 + 9a = 9(5 + a).$$

For our solution with two two-digit numbers we need to take out two single-digit numbers, call them $a$ and $b$, and add them back in the tens place of two of the other numbers. And we get:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 - a - b + 10a + 10b = 9(5 + a + b).$$

For our solution with three two-digit numbers we need to take out three single-digit numbers, call them a, b, and c, and add them back in the tens place of three of the other numbers. And we get:

$$1+2+3+4+5+6+7+8+9-a-b-c+10a+10b+10c = 9(5+a+b+c).$$

So any solution has the form $9k$, where $k$ is some natural number. But $9k$ cannot equal 100 for any natural number $k$, so a combination adding to 100 is impossible.

**Solution 2 (by the organizers).** As noted above only 1- or 2-digit numbers are allowed. Let $k$ be the sum of those digits used in the tens position. Since $1+2+3+4+5+6+7+8+9 = 45$ then Jack's sum is $10k+(45-k) = 9(k+5)$. This is always a multiple of 9 and therefore never equal to 100.

**Solution 3 (by the organizers).** It is known that any natural number is congruent to the sum of its digits modulo 9 (i.e. the number and the sum of its digits leave the same remainder when divided by 9). If $S(n)$ is the sum of the digits of $n$ then this fact is written $S(n) \equiv n (\bmod 9)$. Say that $n_1, n_2, ..., n_k$ are the numbers used by Jack. Then

$$\begin{aligned} 100 &= n_1 + n_2 + ... + n_k \equiv S(n_1) + S(n_2) + ... S(n_k) \\ &= 1+2+3+4+5+6+7+8+9 = 45 \equiv 0 (\bmod 9) \end{aligned}$$

But $100 \equiv 1 \pmod 9$.