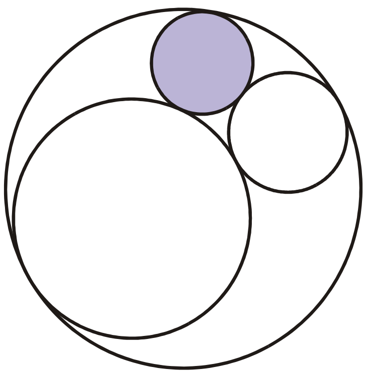
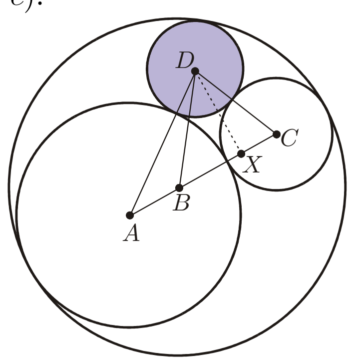
Proposed by Bernardo Ábrego and Silvia Fernández. **March 14-21**



Circles of radii 1 and 2 are externally tangent to each other and internally tangent to a third circle of radius 3. A fourth circle is tangent to the other three circles according to the figure.

What is the radius of this fourth circle?

Solution 1 (by the organizers). Let A, B, C, D be the centers of the circles with radii 2,3,1, and r respectively. The problem asks for the value of r. The circles with centers A and C are externally tangent and internally tangent to the circle of radius 3, thus A, B, and C are collinear. Note that $AC = 3, AB = 1, BC = 2, AD = 2 + r, CD = 1 + r$, and $BD = 3 - r$. Since $BC = 2AB$ then $\text{Area}(BCD) = 2\text{Area}(ABD)$. We calculate both of these areas using Heron's formula: If a triangle has sides a, b, c, and semiperimeter $s = (a+b+c)/2$, then its area equals $\sqrt{s(s-a)(s-b)(s-c)}$.



In ABD the semiperimeter equals $(AB + BD + AD)/2 = 3$, and in BCD the semiperimeter equals $(BC + CD + BD)/2 = 3$. Thus

$$\begin{aligned}\text{Area}(ABD)^2 &= 3(2)(r)(1-r) = 6r(1-r), \text{ and}\\ \text{Area}(BCD)^2 &= 3(1)(2-r)(r) = 3r(2-r).\end{aligned}$$

Then

$$3r(2-r) = \text{Area}(BCD)^2 = 4\text{Area}(ABD)^2 = 24r(1-r)$$

and since $r \neq 0$ then $r = 6/7$.

Solution 2 (by the organizers). We use the same labels as in solution 1, but now we also consider the point X, given as the foot of the perpendicular to AC by D. Instead of using areas, we use Pythagora's Theorem applied to the triangles AXD, BXD, and CXD. Let $DX = h$ and $CX = a$. Then

$$\begin{aligned}(1+r)^2 &= CD^2 = CX^2 + XD^2 = a^2 + h^2,\\ (3-r)^2 &= BD^2 = BX^2 + XD^2 = (2-a)^2 + h^2, \text{ and}\\ (r+2)^2 &= AD^2 = AX^2 + XD^2 = (3-a)^2 + h^2.\end{aligned}$$

Solving for h^2 in the first equation and substituting in the other two we get

$$\begin{aligned}2r - 1 &= a, \text{ and}\\ 3 - r &= 3a.\end{aligned}$$

Solving for a and r we get $a = 5/7$ and $r = 6/7$.