Problem of the Week

Proposed by Bernardo Ábrego and Silvia Fernández. **February 7-14**

A restaurant offers five menus each priced a different whole amount of dollars between \$10 and \$19 inclusive. Last Friday there were 15 couples dining at the restaurant, one couple per table. If every person picked one of the five menus, show that there were at least two tables paying the same amount of money.

Solution (by the organizers). We first count the number of choices each table has of picking their food, given that there there are 5 different menus available. If the two persons at a table ask for different menus then they have 10 choices (these are the number of combinations of two different menus chosen from five possibilities). If the two persons ask for the same menu then they have 5 choices. In total there are 15 choices for each couple.

If two or more couples chose the same pair of menus then clearly they will pay the same amount of money. Thus we may assume that each couple gets a different choice of menus compared to the choices of the other fourteen couples. Moreover, since there are exactly 15 choices, each possible pair of menus is requested by a couple.

Now we must argue that at least two of the fifteen different pairs of menus have the same price sum. Suppose that the prices of the five menus offered by the restaurant are p_1, p_2, p_3, p_4, and p_5. We claim that it is enough to show there are two different ordered pairs $(p_i, p_j), (p_l, p_m)$ with $p_i < p_j, p_l < p_m$ having the same difference. Indeed, if $p_j - p_i = p_m - p_l$ then $p_j + p_l = p_m + p_i$. And by the previous paragraph we know there is a couple asking for menus with prices p_j, p_l and another couple asking for menus priced p_m, p_i (we allow the possibility of $p_j = p_l$). Finally, to prove the claim, observe that all differences $p_j - p_i$ with $p_i < p_j$ are positive integers between 1 and 9 (because the menu prices are integers between 10 and 19). That is, there are 9 possible differences. However there are 10 pairs (p_i, p_j) with $p_i < p_j$ so by the Pigeon-Hole Principle there are at least two pairs with the same difference.