**Problem of the Week.** **May 10-17**

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Let $a$ and $b$ be positive integers such that $a$ divides $b^2$, $b^2$ divides $a^3$, $a^3$ divides $b^4$, $b^4$ divides $a^5$, but $a^5$ does not divide $b^6$. Find with proof a pair $(a,b)$ with this property where $a$ is as small as possible.

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**Solution (inspired by Farzad Ghassemi's solution)**. Let $a = Ad$ and $b = Bd$ with $\gcd(A,B) = 1$. Since $a^3 \mid b^4$ then $b^4 = a^3 q_1$ for some integer $q_1$. Then $d^4B^4 = d^3A^3q_1$ and $dB^4 = A^3q_1$. But $\gcd(A^3,B^4) = 1$ (since $\gcd(A,B) = 1$), thus $A^3 \mid d$. Similarly, since $b^4 \mid a^5$ then $a^5 = b^4q_2$ for some integer $q_2$. Then $dA^5 = B^4q_2$ and , since $\gcd(A^5,B^4) = 1$, then $B^4 \mid d$. Therefore, since $\gcd(A^3,B^4) = 1$, we conclude that $d = A^3B^4D$ for some positive integer $D$. Now, let us look at the last condition:

$$\frac{b^6}{a^5} = \frac{B^6d^6}{A^5d^5} = \frac{B^6A^3B^4D}{A^5} = \frac{B^{10}D}{A^2}, \tag{1}$$

we already know $A^2$ and $B^{10}$ are relatively prime, thus $a^5 \nmid b^6$ if and only if $A^2 \nmid D$. Now let us check that these are sufficient conditions. i.e., we will check that if $a = Ad = A^4B^4D$, $b = Bd = A^3B^5D$ with $\gcd(A,B) = 1$ and $A^2 \nmid D$, then $a \mid b^2 \mid a^3 \mid b^4 \mid a^5 \nmid b^6$. Indeed, $b^2 = (A^2B^6D)\,a$, $a^3 = (A^6B^2D)\,b^2$, $b^4 = (B^8D)\,a^3$, $a^5 = (A^8D)\,b^4$, and the last condition holds since $A^2 \nmid D$ (see (1)).

Finally, to find the smallest value $a = A^4B^4D$ we need that $A \geq 2$ (since $A^2 \nmid D$), $B \geq 1$, and $D \geq 1$. The values $A = 2, B = D = 1$ give $a = 2^4 = 16$ and $b = 2^3 = 8$ which is the pair $(a,b)$ with smallest value $a$. Notice that to find the characterization of all pairs satisfying the conditions we did not use $a \mid b^2$ and $b^2 \mid a^3$.