Problem of the Week #2 **March 15-22, 2004**

Proposed by Bernardo Ábrego and Silvia Fernández.

Find with proof the exact value of the sum

$$\frac{1}{2^1}+\frac{2}{2^2}+\frac{3}{2^3}+...+\frac{n}{2^n}+...$$

Note: If you are familiar with Σ-notation, the sum above can be written as

$$\sum_{n=1}^{\infty}\frac{n}{2^n}.$$

Solution by Takumi Saegusa. Let the partial sum

$$s_m=\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{m}{2^m}.$$

Then, $2s_m=1+\frac{2}{2}+\frac{3}{2^2}+\cdots+\frac{m}{2^{m-1}}$. Subtraction gives

$$2s_m-s_m=(1+\frac{2}{2}+\frac{3}{2^2}+\cdots+\frac{m}{2^{m-1}})-(\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{m}{2^m}),$$

or

$$\begin{aligned} s_m &= 1+\left(\frac{2}{2}-\frac{1}{2}\right)+\left(\frac{3}{2^2}-\frac{2}{2^2}\right)+\cdots+\left(\frac{m}{2^{m-1}}-\frac{(m-1)}{2^{m-1}}\right)-\frac{m}{2^m} \\ &= 1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{m-1}}-\frac{m}{2^m} \text{ (this is a geometric series except for the last term.)} \\ &= \frac{1-\left(\frac{1}{2}\right)^m}{1-\frac{1}{2}}-\frac{m}{2^m}=2-\left(\frac{m+2}{2^m}\right) \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{n}{2^n} &= \lim_{m\to\infty}s_m=\lim_{m\to\infty}\left(2-\left(\frac{m+2}{2^m}\right)\right) \\ &= 2-\lim_{m\to\infty}\frac{m}{2^m}-\lim_{m\to\infty}\frac{1}{2^{m-1}} \\ &= 2-\lim_{m\to\infty}\frac{1}{(\ln 2)\,2^m}-0 \text{ (by L'hospital's rule, since } m\to\infty, 2^m\to\infty \text{ as } m\to\infty) \\ &= 2-0=2. \end{aligned}$$