Problem of the Week #1 — March 8-15, 2004

Proposed by Bernardo Ábrego and Silvia Fernández.

Let n be a positive integer. A group of n^2 people is divided into smaller groups according to the following procedure: Each person is assigned a different number from 1 to n^2. The first group consists of all people whose number is a perfect square. After removing this group, the remaining people are renumbered starting from 1 again. The second group consists of all people whose new number is a perfect square. The process of renumbering the people and removing the group of perfect squares is repeated until one person is left. This person is the only member of the last group. How many groups were formed?

Solution by Takumi Saegusa. I claim that the number of groups formed is $2n-1$. I prove it by mathematical induction.
The basis step.

For $n=1$, the number of groups formed is 1, and clearly, my claim holds for $n=1$.
The inductive step.

Assume that my claim holds for $n=k$ where k is a positive integer. That is, the number of groups formed is $2k-1$ for k^2 people.

Consider the case where there are $(k+1)^2$ people. After assigning the numbers to the people, there are $k+1$ people whose numbers are perfect squares, $1^2, 2^2, \ldots, k^2, (k+1)^2$. From these people, the first group is formed. Then there are $(k+1)^2 - (k+1) = k^2 + 2k + 1 - k - 1 = k^2 + k$ people left. After renumbering, there are k people whose numbers are perfect squares, $1^2, 2^2, \ldots, k^2$, since $k^2 < k^2 + k < (k+1)^2$. From these people, the second group is formed. Then, there are $k^2 + k - k = k^2$ people left. By assumption $2k-1$ groups are formed from these people. Thus, $2 + 2k - 1 = 2k + 1 = 2(k+1) - 1$ groups are formed in total. Thus, my claim holds for $n = k+1$.

Therefore, the number of groups formed is $2n-1$.