Problem of the Week. **May 10-17**

Proposed by Bernardo Ábrego and Silvia Fernández.

Let a and b be positive integers such that a divides b^2, b^2 divides a^3, a^3 divides b^4, b^4 divides a^5, but a^5 does not divide b^6. Find with proof a pair (a, b) with this property where a is as small as possible.

Solution (by organizers). The pair (a, b) with smallest a is $(16, 8)$. First note that $a = 2^4 \mid 2^6 = b^2$, $b^2 = 2^6 \mid 2^{12} = a^3$, $a^3 = 2^{12} \mid 2^{12} = b^4$, $b^4 = 2^{12} \mid 2^{20} = b^5$, but $b^5 = 2^{20} \nmid 2^{24} = a^6$.

Now, suppose (a, b) is the pair with smallest a satisfying the conditions $a^3 \mid b^4$ and $a^5 \nmid b^6$. By the Fundamental Theorem of Arithmetic we may assume that

$$a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} \text{ and } b = p_1^{\beta_1} p_2^{\beta_2} \cdots p_r^{\beta_r}$$

where the p_is are distinct primes and $\alpha_i, \beta_i \geq 0$ for $i = 1, 2, \ldots, r$.

The condition $a^3 \mid b^4$ implies that $3\alpha_i \leq 4\beta_i$ for every $i = 1, 2, \ldots, r$. Similarly, $a^5 \nmid b^6$ implies that $5\alpha_j > 6\beta_j$ for some $1 \leq j \leq r$. Let $A = 2^{\alpha_j}$ and $B = 2^{\beta_j}$, observe that $A^3 \mid B^4, A^5 \nmid B^6$, and $a \geq A$. Thus we may assume that $a = A = 2^{\alpha_j} = 2^{\alpha}$ and $b = B = 2^{\beta_j} = 2^{\beta}$ with $6/5 < \alpha/\beta \leq 4/3$. The smallest numerator α of all fractions α/β in the range $(6/5, 4/3]$ is precisely $\alpha = 4$ (with $\beta = 3$). Therefore $(a, b) = (2^4, 2^3) = (16, 8)$ is the pair with smallest a satisfying $a^3 \mid b^4$ and $a^5 \nmid b^6$ and, as we saw before, it also satisfies that $a \mid b^2$, $b^2 \mid a^3$, and $b^4 \mid a^5$.