Problem of the Week 5, Fall 2008

Real numbers a, b, and c satisfy that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}.$$

Prove that, for every odd integer n, the following identity holds

$$\left(\frac{1}{a+b+c}\right)^n=\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}=\frac{1}{a^n+b^n+c^n}.$$

Solution by organizers. Given

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c},$$

we have

$$\begin{aligned}(a+b+c)(bc+ac+ab) &= abc,\\ 2abc+a^2c+a^2b+b^2c+ab^2+ac^2+bc^2 &= 0,\\ (a+b)(b+c)(a+c) &= 0.\end{aligned}$$

This means that either $b=-a, c=-a$, or $c=-b$. Since a, b, and c play the same role in the identities above, we can assume $c=-b$. Thus, for every odd integer n, $c^n=-b^n$, $\frac{1}{c^n}=-\frac{1}{b^n}$, and

$$\frac{1}{a^n}=\left(\frac{1}{a+b+c}\right)^n=\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}=\frac{1}{a^n+b^n+c^n}.$$