Problem of the Week 9, Fall 2005

Solution by Andrew Jones. First we show 2005 cannot be written as the sum of numbers each of them equal to 119 or 18. That is, it is not possible to write 2005 = 119A+18B where A and B are integers, $A \geq 0$ and $B \geq 0$.

<u>proof (by contradiction)</u>
Assume 2005=119A+18B for some A and B are integers, $A \geq 0$ and $B \geq 0$. (1)
For (1) to hold we need A< 16 because if A> 17 then 119A+18B > 119(17) = 2023>2005.
This means that A equals 0, 1, 2, 3,…, 15, or 16. We can go through all of these possibilities and disprove them all.

For A = 0, Equation (1) gives 2005 = 18B. But B=2005/18 is not an integer. Thus $A \neq 0$.
For A = 1, Equation (1) gives B = 1886/18, not an integer. Thus $A \neq 1$. Similarly for A=2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 and 16, Equation (1) gives B = 1767/18, 1648/18, 1529/18, 1410/18, 1291/18, 1172/18, 1053/18, 934/18, 815/18, 696/18, 577/18, 458/18, 339/18, 220/18, and 101/18. But all these values of B are not integers so Equation (1) never holds.

Now we will prove that for any integer $n \geq 2006$ we can write n in the form n=119A+18B where A and B are non-negative integers.

<u>proof (by Mathematical Induction)</u>
P(n) : n = 119A+18B where A and B are integers and $A \geq 0$ and $B \geq 0$.

<u>Basis Step</u>
P(2006) is true: 2006 = 119(4) + 18(85)

<u>Inductive Step</u>
We assume P(n) is true for some $n \geq 2006$. That is, n = 119A+18B for some $A \geq 0$ and $B \geq 0$. (2)
We want to prove that P(n+1) is true. That is, n+1 = 119C + 18D for some $C \geq 0$ and $D \geq 0$. This is done by cases and we will use the fact that we can add and/or subtract 119's and 18's as long as we are left with a positive number of 119's and 18's.

<u>Case 1</u>
Note that 1 = 119(5) + 18(-33). (3)
If $B \geq 33$ then adding (2) and (3) gives n+1 = 119(A+5) + 18(B-33) where $A+5 \geq 0$ and $B-33 \geq 0$.

<u>Case 2</u>
Note that 1 = 119(-13) + 18(86). (4)
If $B \leq 32$ then $A \geq 13$ or $A \leq 12$. But $A \leq 12$ and $B \leq 32$ gives $n \leq 119(12) + 18(32) = 2004$ which is a contradiction. Thus $A \geq 13$. Similarly to Case 1, adding (2) and (4) gives
n+1 = 119(A-13) + 18(B+86) where $A-13 \geq 0$ and $B+86 \geq 0$.

In both cases we proved P(n+1) to be true. Therefore by Mathematical Induction P(n) is true for all $n \geq 2006$.