Problem of the Week 12, Fall 2005

Solution #1 by the organizers. If $x+y=0$ then $y=-x$ and thus

$$\begin{aligned}\left(\sqrt{1+x^2}+x\right)\left(\sqrt{1+y^2}+y\right) &= \left(\sqrt{1+x^2}+x\right)\left(\sqrt{1+(-x)^2}+(-x)\right)\\ &= \left(\sqrt{1+x^2}+x\right)\left(\sqrt{1+x^2}-x\right)\\ &= 1+x^2-x^2=1.\end{aligned}$$

Now assume that

$$\left(\sqrt{1+x^2}+x\right)\left(\sqrt{1+y^2}+y\right)=1.$$

Multiplying both sides by $\sqrt{1+y^2}-y$

$$\begin{aligned}\left(\sqrt{1+x^2}+x\right)\left(\sqrt{1+y^2}+y\right)\left(\sqrt{1+y^2}-y\right) &= \sqrt{1+y^2}-y\\ \sqrt{1+x^2}+x &= \sqrt{1+y^2}-y.\end{aligned}$$

This gives

$$x+y=\sqrt{1+y^2}-\sqrt{1+x^2}.$$

Now square both sides and simplify

$$\begin{aligned}x^2+y^2+2xy &= 1+y^2+1+x^2-2\sqrt{1+y^2}\sqrt{1+x^2}\\ \sqrt{\left(1+y^2\right)\left(1+x^2\right)} &= 1-xy.\end{aligned}$$

Finally square both sides again and regroup

$$\begin{aligned}1+y^2+x^2+y^2x^2 &= 1-2xy+x^2y^2\\ y^2+x^2+2xy &= 0\\ (x+y)^2 &= 0\\ x+y &= 0.\end{aligned}$$

Solution #2 by the organizers. Follow the previous proof up to

$$\sqrt{1+x^2}+x=\sqrt{1+y^2}-y. \tag{1}$$

Let $f(x)=\sqrt{1+x^2}+x$. Then the derivative of f satisfies

$$f'(x)=\frac{x}{\sqrt{1+x^2}}+1=\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}>\frac{x+|x|}{\sqrt{1+x^2}}\geq 0.$$

Since $f'(x)>0$ for all x then$f$ is strictly increasing and hence one-to-one. Since $f(x)=f(-y)$ from (1) then $x=-y$, that is, $x+y=0$.