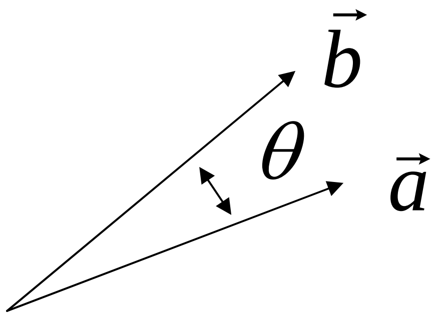
Review: Vector Algebra And Analysis

Scalar and Vector Products

For two vectors:

$$\vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k}$$

$$\vec{b} = b_x \vec{i} + b_y \vec{j} + b_z \vec{k}$$



and θ is the angle between $\vec{a}$ and $\vec{b}$.

The magnitudes of $\vec{a}$ and $\vec{b}$ are:

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}, \qquad b = \sqrt{b_x^2 + b_y^2 + b_z^2},$$

R1.1 Scalar product

$$\vec{a} \cdot \vec{b} = ab\cos\theta \qquad (1.1)$$

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z \qquad (1.2)$$

R1.2 Vector product

$$\left|\vec{a}\times\vec{b}\right| = ab\sin\theta \qquad (2.1)$$

$$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a} \qquad (2.1)$$

The direction of $\vec{a}\times\vec{b}$ is determined by the Right-Hande-Rule.

$$\vec{a}\times\vec{b} = (a_x\vec{i} + a_y\vec{j} + a_z\vec{k})\times(b_x\vec{i} + b_y\vec{j} + b_z\vec{k})$$
$$= (a_y b_z - a_z b_y)\,\vec{i} + (a_z b_x - a_x b_z)\vec{j} + (a_x b_y - a_y b_x)\vec{k} \qquad (2.3)$$

The above equation can also be expressed as :

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \qquad (2.4)$$

1.3 Gradient (Grad) and del Operators

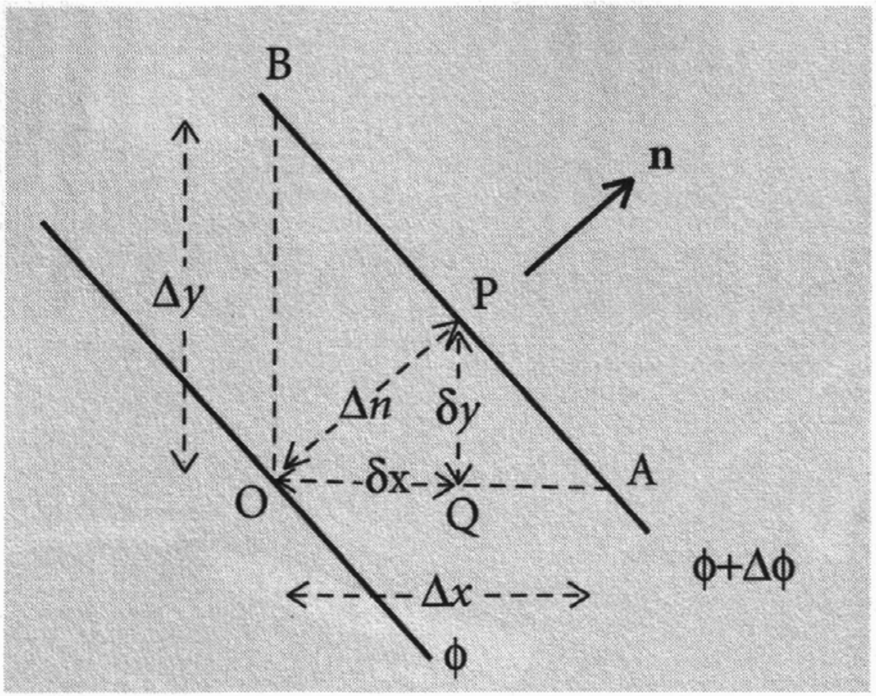


Fig.3 Two close contours in a scalar field in the x-y coordinate system.

Gradient (Grad) and del Operators

∇ is a Vector Operator:

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

$$\text{Grad } \phi = \nabla \phi = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\phi$$

$$= \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}$$

The gradient is the maximum rate of change of the scale field ϕ, which is perpendicular to the contour at that point.

Physical Example: Electric potential and electric field.

1.4 Divergence (Div)

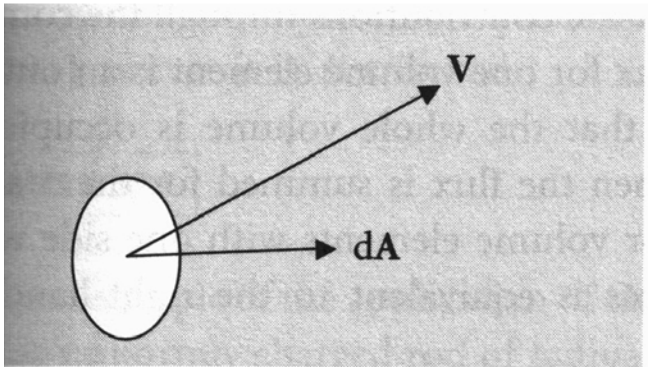


Fig. 5. The relationship of $\vec{V}$ and the small vector area $\vec{A}$

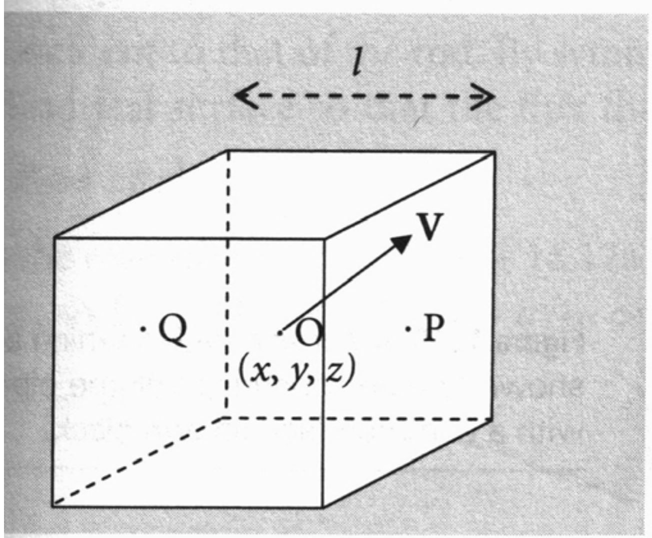


Fig. 6. The cube box surrounding the point P, and the vector field $\vec{V}$.

Divergence (Div)

$$\text{Div}\ \vec{V} = \nabla \cdot \vec{V} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(V_x\vec{i} + V_y\vec{j} + V_z\vec{k}\right)$$

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial x} + \frac{\partial V_z}{\partial x} \qquad (4.6)$$

The divergence is the outward flux per unit volume at that point.
It is a scalar quantity.

Physical Example: electric flux per unit volume.

Divergence Theorem

$$\oint_{S_V} \vec{V} \cdot d\vec{s} = \int_V (\nabla \cdot \vec{V}) dv$$

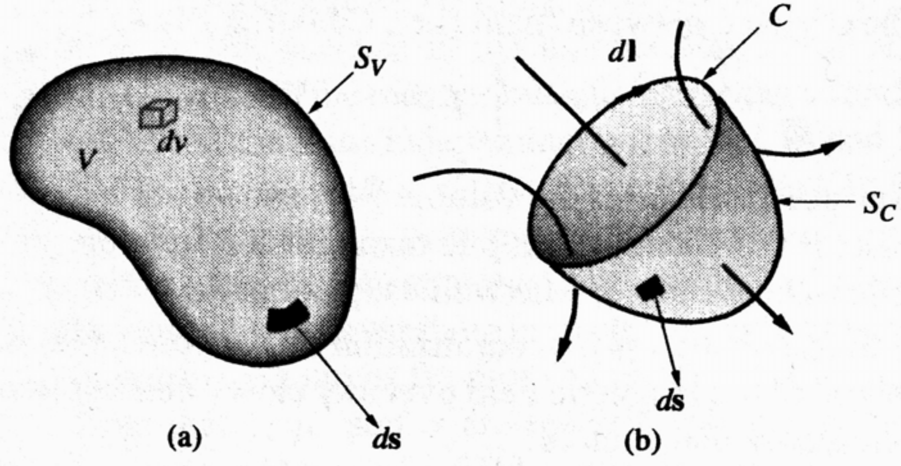


FIGURE 1.1. Contour, surface, and volume. (a) A closed surface S_V enclosing a volume V. (b) A closed contour C enclosing a surface S_C.

1.5 Curl of a Vector Field

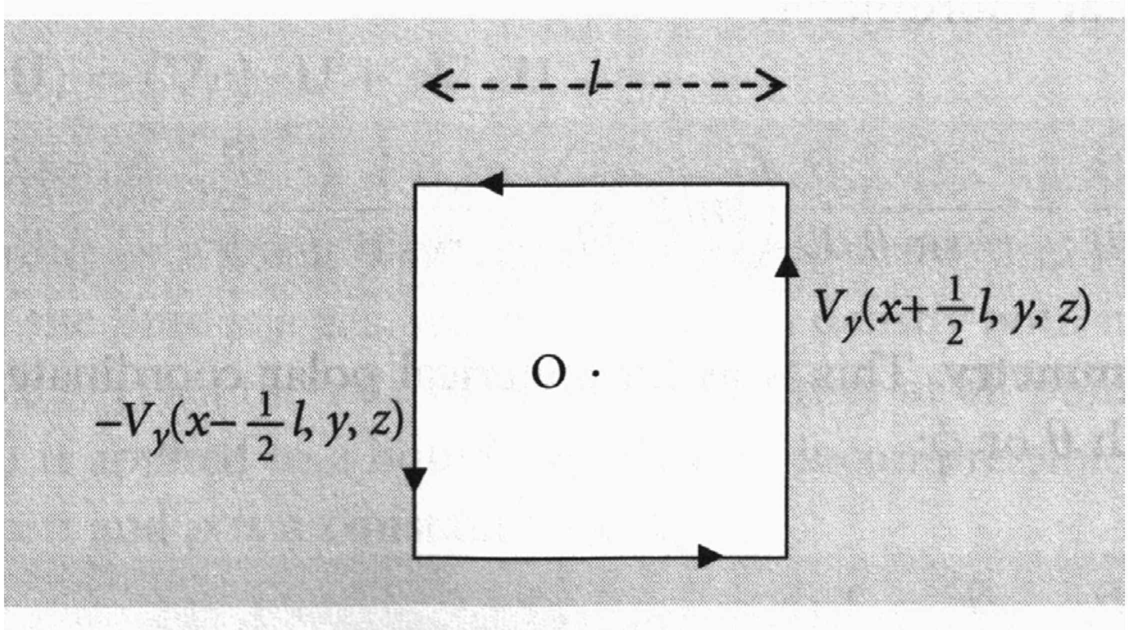


Fig. 6. The circulation in a square contour in the x-y plane.

Curl of a Vector Field

$$\text{Curl } \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\vec{i} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\vec{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\vec{k} \qquad (5.2)$$

The Z components of curl $\vec{V}$ is the circulation $\int_c \vec{V} \cdot d\vec{s}$ per unit area in the x-y plane.

In a x-y-z coordinate, the vector field $\vec{V}$ has a rotatory component in a plane whose normal is in the direction of $\vec{V}$. The circulation per unit area in that plane is given by $\left|\nabla \times \vec{V}\right|$

1.6 Laplacian Operator

Laplacian operator is a scalar operator, and is defined as :

$$\nabla^2 = \nabla \cdot \nabla$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad (6.1)$$

1.7 Other operators

For a vector field $\vec{V} = V_x\vec{i} + V_y\vec{j} + V_z\vec{k}$, we can prove that

$$\nabla \times \nabla \times \vec{V} = \nabla(\nabla \cdot \vec{V}) - \nabla^2\vec{V} \qquad (7.1)$$