ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section, and (5) a grading table. You must work all the problems on the exam. Show ALL your work in your bluebook and **BOX IN YOUR FINAL ANSWERS**. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Textbooks, classnotes, crib sheets, or calculators are NOT permitted.

1. (40 points) Let $A$ be a nonsingular $n \times n$ real matrix and $b$ a vector in $\mathbf{R}^n$. Let $M, N$ be two $n \times n$ real matrices obtained through the following splitting

$$A = M - N$$

a. Show that the iterative process given by

$$\begin{cases} M\,x^{k+1} = N\,x^k + b\,; \quad k \geq 0 \\ x^0 \in \mathbf{R}^n \quad \text{(given)} \end{cases}$$

is equivalent to the following

$$\begin{cases} \text{Given } x^k\,, \\ \text{compute} \quad r^k = b - A\,x^k \\ \text{solve} \quad Q\,z^k = r^k \\ \text{define} \quad x^{k+1} = x^k + z^k \end{cases}$$

where the matrix $Q$ is to be found.

b. Prove that for all $k \geq 0$, we have

$$\begin{cases} r^{k+1} = (I - A\,Q^{-1})\,r^k \\ z^{k+1} = (I - Q^{-1}\,A)\,z^k \end{cases}$$

c. Let $||\ ||$ be a matrix nom subordinate to a vector norm. Prove that if the number $\delta = ||M^{-1}\,N||$ is less than 1, then

$$||x^k - x|| \leq \frac{\delta}{1-\delta}\,||x^k - x^{k-1}||$$

where $x$ satisfies $A\,x = b$.

d. Find the explicit form for the iteration matrix $M^{-1}\,N$ in the Gauss-Seidel method.

e. Assume that $A$ is a strictly diagonally dominant matrix. Prove that Gauss-Seidel iterate sequence $(x^k)$ must converge for any starting point $x^0$.

2. (40 points) Count the number of multiplications and/or divisions needed to invert a unit $n \times n$ lower triangular matrix.

3. (30points) Describe the iterates $x^k$ constructed with the preconditionned conjugate gradient algorithm for solving the linear system $A\,x = b$.

4. (30 points)

   Let $A$ and $B$ be two $n \times n$ real matrices and $||\cdot||$ a matrix norm subordinate to a vector norm. Show that if $||AB - I|| = r < 1$, then

$$||A^{-1} - B|| \;\leq\; \frac{r}{1-r}||B||$$

5. (30 points) Let $A$ and $\delta\,A$ be two real and symmetric $n \times n$ matrices. Let $\alpha_l$ (resp. $\beta_l$), $(l = 1, ..n)$ be the eigenvalues of $A$ (resp. of $A + \delta\,A$) counted with their multiplicity.

   a. Explain why we can assume that

$$\alpha_1 \leq \alpha_2 \leq ... \leq \alpha_n$$

   and

$$\beta_1 \leq \beta_2 \leq ... \leq \beta_n$$

   b. Prove that

$$|\beta_l - \alpha_l| \leq ||\delta\,A||_2 \quad 1 \leq l \leq n$$

6. (30 points) Consider the following Householder transformation

$$H_u \;=\; I - 2\,u\,u^T \;; \quad \forall u \in \mathbf{R}^n$$

   a. Prove that if $||u||_2 = 1$, then $H_u$ is an orthogonal matrix.

   b. Prove that for any $a \in \mathbf{R}^n$, there is $u \in \mathbf{R}^n$ and $\alpha \in \mathbf{R}$ such that

$$H_u\,a \;=\; \alpha\,e^{(1)}$$

   where $e^{(1)}$ is the first vector of the standard basis of $\mathbf{R}^n$.

   c. Describe the $QR$ factorization process when using successive Householder transformations.