## Midterm 2 Review Questions

1. Show that
$$\int_{|z|=1} z^\alpha \, dz = \begin{cases} \frac{e^{2\pi i\alpha}-1}{\alpha+1}, & \alpha \neq -1 \\ 2\pi i, & \alpha = -1. \end{cases}$$
for any $\alpha \in \mathbb{C}$, where $z^\alpha$ is such that $1^\alpha = 1$ and the contour begins at $z = 1$.

2. Assuming that none of the points 0, $i$ and $-i$ lie on the closed contour $C$ calculate all possible values of the integral
$$\int_C \frac{dz}{z(1+z^2)}$$
depending on the location of the contour $C$.

3. Evaluate the integral
$$\frac{1}{2\pi i}\int_C \frac{ze^z}{(z-a)^3}\, dz$$
given that the point $a$ is located inside $C$.

4. (Morera's Theorem) Suppose that $f(z)$ is continuous in a region $U$ and for every rectangle with sides parallel to the coordinate axes contained in $U$
$$\int_\Gamma f(z)\, dz = 0,$$
where $\Gamma$ denotes the boundary of the rectangle. Show that $f$ is analytic in $U$.

5. Suppose that a series $\sum_{j=0}^\infty f_j(z)$ of analytic functions over a region $U$ converges uniformly for $z$ in $U$.

   (a) Show that the sum of the series $f(z)$ is analytic in $U$.

   (b) Show that the derivatives of $f$ can be obtained by
$$f^{(k)} = \sum_{j=0}^\infty f_j^{(k)}$$

   (c) If each $f_j$ has the form $\sum\limits_{n=0}^\infty a_{nj}(z-z_0)^n$ then
$$f(z) = \sum_{n=0}^\infty a_n (z-z_0)^n,$$
   where $a_n = \sum\limits_{j=0}^\infty a_{nj}$.

*Hints: (a) Use the result of the previous problem (or any version of Morera's Theorem). (b) Use Cauchy's Integral Formula and its extensions for the derivatives. (c) Use that $a_n = \frac{1}{n!} f^n(z_0)$.]*

6. Let $f(z)$ be an entire function, with $|f(z)| \leq C|z|$ for all $z$, where $C$ is a constant. Show that $f(z) = Az$, where $A$ is a constant.

7. Suppose that $f$ is an entire function with $f(z) = f(iz)$ for all $z$. Prove that $g(z) = f(\sqrt[4]{z})$ is entire, and
$$\frac{g^{(n)}(0)}{n!} = \frac{f^{(4n)}(0)}{(4n)!}.$$

8. Suppose $u$ is harmonic in a region $U \subseteq \mathbb{C}$ and $v$ is a harmonic conjugate of $u$.
   (a) Show that $v$ is unique up to an additive constant.
   (b) Show that $-u$ is a harmonic conjugate of $v$.

9. Verify that the following functions are harmonic, and find their harmonic conjugates:
   (a) $u(x, y) = e^{-y} \sin x$
   (b) $u(x, y) = x^3 y - xy^3$
   (c) $u(x, y) = \arctan(x/y)$, $y > 0$.

10. Suppose $U \subseteq \mathbb{R}^2$ is a region and $C$ is a closed simple contour contained in $U$ with its interior.
   (a) If $u$ is harmonic in $U$ show that $\int_C \frac{\partial u}{\partial n}\, ds = 0$.
   (b) If $g$ is harmonic in $U$ and $\frac{\partial g}{\partial n} = 0$ on $C$ then $g$ is constant inside $C$.

11. Given the function $f(z) = \dfrac{1}{z(1+z^2)}$ expand it in a Laurent series into powers of $z$ in the regions
   (a) $|z| < 1$
   (b) $|z| > 1$.

12. Determine all singularities of the functions in $\overline{\mathbb{C}}$, classify them, and find principal parts of Laurent series about each singularity that admits a Laurent expansion:
   (a) $\dfrac{e^{2z} - 1 - 2z}{z^4}$
   (b) $\dfrac{z^{1/3} - 1}{z - 1}$.
   (c) $\dfrac{1}{\sin z}$.
   (d) $e^{-1/z^2}$.

13. Suppose $f$ is such that the only singularities of $f$ in $\overline{\mathbb{C}}$ (including a possible one at $z = \infty$) are poles. Show that $f(z)$ must be a rational function of $z$.

14. Suppose $f(z)$ is analytic for $0 < |z - z_0| < \rho$ except for a sequence of poles $z_n \to z_0$. Let $U = \{z \in \mathbb{C} : 0 < |z - z_0| < \rho,\ \forall n\ z \neq z_n\}$. Show that $f(U)$ is dense in $\mathbb{C}$.

15. Determine all singular points of the functions, and describe the regions in which analytic continuations can be obtained. In which case(s) are analytic continuations unique?

    (a) $\sum\limits_{n=0}^{\infty}(-1)^n z^{2n}$, $|z| < 1$

    (b) $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} z^{2n-1}$, $|z| < 1$

16. Show that the function $f(z) = \sum\limits_{n=0}^{\infty} z^{n!}$ has the unit circle $|z| = 1$ as its natural boundary.

17. Evaluate the contour integral using binomial expansion and Cauchy's Integral Formula:

$$\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z}.$$

    Use this result to establish the following real formula:

$$\frac{1}{2\pi} \int_0^{2\pi} \cos^{2n}\theta \, d\theta = \frac{(2n)!}{4^n (n!)^2}$$

18. Use techniques of complex intergration to evaluate:

    (a) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(1 + x^2)^2}\, dx$

    (b) $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\, dx$