Midterm 1 Review Questions

1. Sketch the sets in the complex z-plane described by the equations

   (a) $|z|^2 + z + z^* = 0$

   (b) $|z-1| = |z-2|$

   (c) $|z-2| - |z+2| > 5$

   (d) $\mathrm{Im}\left(\frac{z-z_1}{z-z_2}\right) = 0.$

2. If $z^*w \neq 1$ prove that

$$\left|\frac{z-w}{1-z^*w}\right| < 1, \quad \text{if} \quad |z|<1 \ \text{and} \ |w|<1$$

   and

$$\left|\frac{z-w}{1-z^*w}\right| = 1, \quad \text{if} \quad |z|=1 \ \text{or} \ |w|=1.$$

3. Prove that

$$|(1+i)z^3 + iz| < \frac{3}{4} \quad \text{if} \quad |z| < \frac{1}{2}.$$

4. Prove that if $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$ then the points z_1, z_2, z_3 are vertices of an equilateral triangle inscribed in the unit circle $|z| = 1$.

5. Let

$$f(z) = \lim_{n\to\infty} \frac{z^n}{1+z^n}.$$

   (a) What is the domain of definition of $f(z)$, that is for which complex numbers z does the limit exist?

   (b) Give an explicit formula for $f(z)$.

   (c) Is the convergence uniform on the domain of f? On what region(s) of $\mathbb{C}$ is the convergence uniform?

6. Consider the series

$$\sum_{n=0}^{\infty} \tfrac{1}{(n+1)!}\, z^n$$

   (a) Find the radius of convergence.

   (b) Describe the region(s) in the complex plane in which the convergence is uniform.

   (c) Compute explicitly the sum of the series for all values z for which the series converges.

7. Give an example of a power series $\sum a_n z^n$ whose radius of convergence is 1, and which converges uniformly in the unit disk $|z| < 1$.

8. Using the real-variable Taylor series $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots$ for $|x| < 1$ and the formal algebra of series, prove that the complex series

$$z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \ldots$$

converges for all $|z| < 1$ to the principal value of $\ln(1+z)$. (See p. 17 in the textbook.)

9. Find the fractional linear transformation which carries $0,\ i,\ -i,\ -\frac{i}{2}$ into $1,\ -1,\ 0,\ \frac{1}{5}$.

10. Find a fractional linear transformation that carries the circles $|z| = 2$ and $|z-1| = 4$ into concentric circles.

11. Suppose that a linear fractional transformation carries one pair of concentric circles into another pair of concentric circles. Prove that the ratios of the radii must be the same.

12. Show that the most general fractional linear transformation that transforms the open unit disk $|z| < 1$ onto itself has the form

$$S(z) = e^{i\theta}\frac{z-a}{1-za^*}$$

where $\theta \in \mathbb{R}$ and $|a| < 1$.

13. For each of the following regions R, determine if it is possible to define a continuous branch of the logarithm function on R. Justify your answer by either constructing a continuous branch of logarithm on R, or proving that none exists.

    (a) $R = \{z : 1 < |z| < 2\} \setminus \{z = x + iy : x = y = t,\ t \in (\frac{1}{\sqrt{2}}, \sqrt{2})\}$

    (b) $R = \{z = x + iy : z \neq 0,\ \max(|x|, |y|) < 1\}$.

14. If $f(z)$ is holomorphic in a region R and $|f(z)|$ is constant, prove that $f(z)$ is constant.

15. Consider the function

$$f(z) = \begin{cases} z^5|z|^{-4}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

Show the the Cauchy-Riemann equations hold for the real and imaginary parts of f at $z = 0$, however $f'(0)$ does not exist. Under what additional conditions do the Cauchy-Riemann equations imply complex differentiability of a complex function $f(z)$?

16. Express the function $w = \arcsin(z)$ through the complex logarithm, and determine all possible w for each z. Show that

$$\frac{d}{dz}\arcsin(z) = \frac{1}{\sqrt{1-z^2}}.$$

(There are infinitely many branches for arcsin and two branches for the square root. Investigate the relation between the branches of arcsin and its derivative. See pp. 27-28 in the textbook.) Same question for $\arccos(z)$ and $\operatorname{arcsinh}(z)$.

17. Find the integral

$$\int_C \cos(z)\, dz$$

from the origin to the point $1+i$ taken along the parabola $y = x^2$ in the complex plane.

18. Find the integrals over the positively oriented unit circle C:

    (a) $\displaystyle\int_C \bar{z}\, dz$

    (b) $\displaystyle\int_C \frac{1}{z-\frac{1}{2}}\, dz$

    (c) $\displaystyle\int_C \frac{e^{2z}+e^z-6}{z^2+z-6}\, dz$