May 12, 2019 MATH 655 Prof. V. Panferov

Final Exam Review Questions

1. If $f'(z)$ exists for every z in an open set $U \subseteq \mathbb{C}$ prove that the derivative of f of any order exists for every z in U.

2. If $f(z)$ is analytic and non-constant for z in a region U prove that it is impossible for the zeros of f to have an accumulation point in U.

3. If $f(z)$ is analytic and non-constant for z in an open set U prove that $f(U)$ is open. *Hint: If $f(z_0) = w_0$ then the analytic function $g(z) = f(z) - w_0$ has a zero at z_0 which must be isolated (why?). Thus, for $\varepsilon > 0$ small enough the real-valued function $|g(z)|$ has a positive minimum on the circle $|z - z_0| = \varepsilon$. Use Rouché's Theorem to complete the proof.*

4. (a) Find the images of the following sets under the mapping $w = \coth z$:

   (i) $D = \{z : \pi/4 < \mathrm{Im}(z) < 3\pi/4\}$.

   (ii) $E = \{z : \pi/4 < \mathrm{Im}(z) < \pi/2, \mathrm{Re}(z) > 0\}$.

   (b) What family of strips are mapped into the interior of the unit circle by the function $w = \coth z$? By the function $w = \tan z$?

5. Prove that the function $\arctan z$ has only two branch points $\pm i$. *Hint: Recall the definition of a branch point.*

6. Consider the following four types of transformations:

$$w = z + b; \qquad w = 1/z; \qquad w = kz; \qquad w = \frac{az + b}{cz + d}, \quad ad - bc \neq 0.$$

   (Here a, b, c, d, k denote complex constants, and z, w are complex variables.) Show that each transformations takes circles into either circles or lines.

7. Give examples of conformal mappings as follows:

   (a) From $\{z : \mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0\}$ onto $\{z : |z| < 1\}$.

   (b) From $\{z : |z| < 1\}$ onto $\{z : \mathrm{Re}(z) < 0\}$.

   (c) From $\{z : |z| < 1\}$ onto itself, such that $f(0) = i/2$.

   (d) From $\{z : z \neq 0, 0 < \arg(z) < \frac{3\pi}{2}\}$ onto $\{z : z \neq 0, \frac{\pi}{2} < \arg(z) < \pi\}$.

   (e) From $\{z : |z| < 1\} \setminus [0, 1)$ onto $\{z : |z| < 1\}$.

8. A fractional linear transformation maps the annulus $r < |z| < 1$ onto the region bounded by the two circles $|z - 1| < 1$ and $|z - 1/4| < 1/8$. Find r.

9. How many different Laurent expansions into powers of z are there for the function

$$g(z) = \frac{1}{1-z^2} + \frac{1}{3-z} \ ?$$

In which region is each of them valid? Determine the coefficients of the Laurent series for each of the cases explicitly.

10. Given the function element (f, D)

$$f(z) = 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \cdots + (-1)^{n-1}\frac{(2n-3)!!}{2^n\, n!}z^n + \ldots, \qquad D = \{z : |z| < 1\}$$

find the Taylor series for its analytic continuation into the region $E = \{z : |z-1| < 1\}$. Is this analytic continuation unique? *Hint: The coefficients of the series are binomial coefficients* $\binom{p}{n}$ *for a certain* p.

11. Evaluate the integral

$$\int_C \frac{e^z - 1}{z^2(z-1)}\, dz$$

where C is a closed contour that has winding numbers 1 about $z = 0$ and -2 about $z = 1$. Sketch an example of a contour C.

12. Use techniques of complex intergration to evaluate:

(a) $\displaystyle\int_0^\infty \frac{1}{1+x^{100}}\, dx$ (b) $\displaystyle\int_{-\infty}^\infty \frac{\sin^2 x}{x^2}\, dx$