Wed, Mar 27, 2019 **MATH 655** Quiz 2

Name: (print) Solutions.

Each problem is worth 2 points. Show all your work.

1. Evaluate the contour integrals around the positively oriented unit circle:

(a) $\displaystyle\int_{|z|=1} \frac{\sin z}{2z+i}\, dz$

$\frac{\sin z}{2z+i}$ is analytic everywhere except $z = -\frac{1}{2}i$

Cauchy I.F. $\Rightarrow$

$$\int_{|z|=1} \frac{\sin z}{2z+i}\, dz = 2\pi i \sin\left(-\frac{i}{2}\right)$$
$$= 2\pi \sinh\left(\frac{1}{2}\right)$$
$$= \pi\left(e^{\frac{1}{2}} - e^{-\frac{1}{2}}\right).$$

(b) $\displaystyle\int_{|z|=1} \cot z\, dz$

$$\cot z = \frac{\cos z}{\sin z} = \frac{\cos z}{z}\,\frac{z}{\sin z},$$

$\frac{z}{\sin z} \to 1$ as $z \to 0$

and $\frac{z}{\sin z}$ is analytic on $|z|<1$

$$\int_{|z|=1} \cot z\, dz = \int_{|z|=1} \frac{1}{z}\cdot \cos z\, \frac{z}{\sin z}\, dz$$
$$= 2\pi i \cos z \left.\frac{z}{\sin z}\right|_{z=0} = 2\pi i.$$

2. If f is an entire function such that $\mathrm{Re}(f) \geq 0$ everywhere prove that f is constant.

Consider $g(z) = e^{-f(z)} = e^{-u(z)} e^{-iv(z)}$
$(f = u + iv)$

$$0 < |g(z)| = e^{-u(z)} \leq 1$$

$g(z)$ is an entire fn and $g(z)$ is bounded

By Liouville $g(z)$ is constant

$$e^{-f(z)} = c > 0$$
$$-f(z) = \ln c$$
$$f(z) = \ln\frac{1}{c} \text{ - constant.}$$

Please turn over...

3. Let Γ be a simple contour in $\mathbb{R}^2$ and U the region inside it. Suppose v and w are solutions the Neumann problem:

$$\begin{cases} \Delta v = 0 & \text{in } U \\ \frac{\partial v}{\partial n} = \psi & \text{on } \Gamma \end{cases} \qquad \begin{cases} \Delta w = 0 & \text{in } U \\ \frac{\partial w}{\partial n} = \psi & \text{on } \Gamma \end{cases}$$

where ψ is continuous on Γ and such that $\int_\Gamma \psi \, ds = 0$. Prove that $w = v + C$ where C is a constant.

Let $g = w - v$. Then

$$\begin{cases} \Delta g = 0 & \text{in } U \\ \frac{\partial g}{\partial n} = 0 & \text{on } \Gamma. \end{cases}$$

Consider the corresponding fn. $h(z)$ s.t. $f(z) = h(z) + i g(z)$ is analytic ($-h(z)$ is the harmonic conjugate of $g(z)$).

Then $\begin{cases} \Delta h = 0 & \text{in } U \\ h = h_0 & \text{on } \Gamma \end{cases}$ (constant)

Since $\frac{\partial h}{\partial s} = -\frac{\partial g}{\partial n} = 0$ on Γ.

By uniqueness of Dirichlet, $h = h_0$ in U

Therefore $g(z) = g_0$ = constant in U.