Wed, Feb 13, 2019 **MATH 655** **Quiz 1**

Name: (print) Solutions.

Each problem is worth 2 points. Show all your work.

1. For what $z \in \mathbb{C}$ does the series converge (diverge)? In what region(s) of $\mathbb{C}$ does the series converge uniformly?

$$\sum_{n=1}^{\infty} \frac{z^n}{z^2+n^2} = \sum_{n=1}^{\infty} a_n$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{z^{n+1}}{z^2+(n+1)^2}\,\frac{z^2+n^2}{z^n}\right| = \left|\frac{z^2+n^2}{z^2+(n+1)^2}\right||z| \xrightarrow[n\to\infty]{} |z|$$

The series converges for $|z|<1$ and diverges for $|z|>1$.

For $|z|\leq 1$, $\left|\frac{z^n}{z^2+n^2}\right| \leq \frac{1}{n^2}$ (forms convergent series)

so by comparison, the series also converges for $|z|=1$, and by Weierstrass, the series converges uniformly for $|z|\leq 1$.

2. How many points can coincide with their images in a (non-identity) fractional linear transformation $w=\dfrac{az+b}{cz+d}$?

Solve $z=\dfrac{az+b}{cz+d} \Rightarrow cz^2+dz=az+b$

$cz^2+(d-a)z-b=0$

If $c=0$ this is a linear eqn that has 1 solution $z=\frac{b}{d-a}$

($d-a\neq 0$ since it's non-identity)

If $c\neq 0$ this is a quadratic eqn that has 2 solutions $z=\frac{a-d}{2c}\pm\sqrt{\left(\frac{d-a}{2c}\right)^2+\frac{b}{c}}$

if $\left(\frac{d-a}{2c}\right)^2+\frac{b}{c}\neq 0$

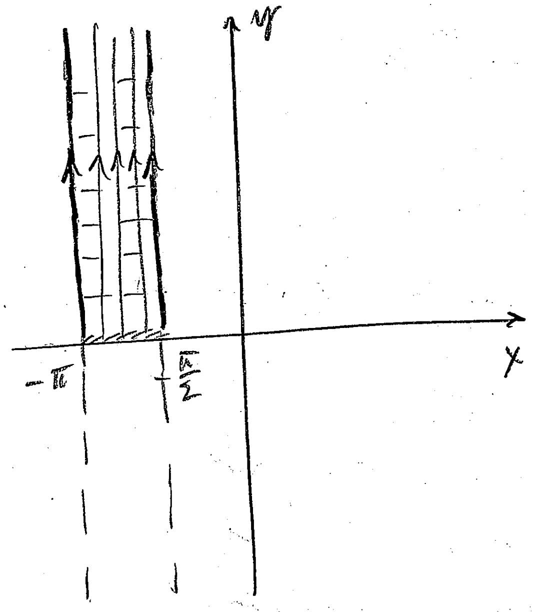
and 1 solution otherwise.

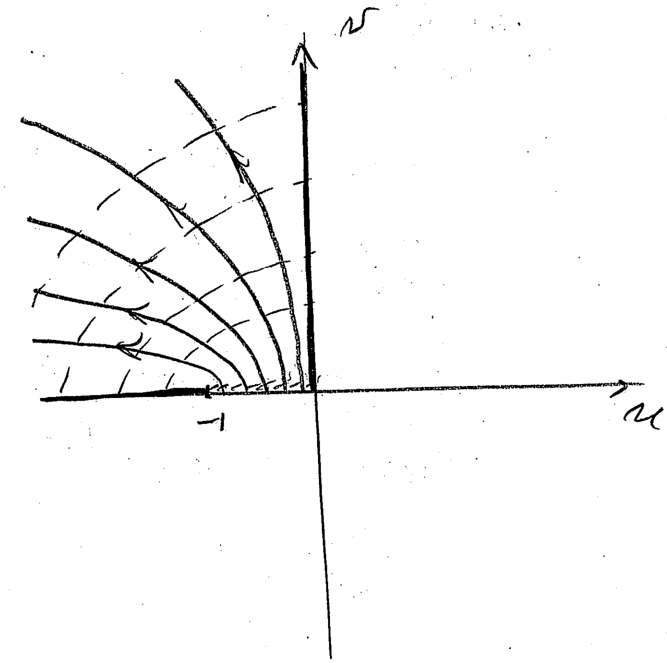
If $w=\dfrac{az+b}{cz+d}=$ constant ($ad-bc=0$) then there is 1 solution $w=z$ as well.

Please turn over...

3. Find the image of the region $\{z \in \mathbb{C} : -\pi \leq \mathrm{Re}(z) \leq -\frac{\pi}{2},\ \mathrm{Im}(z) \geq 0\}$ by the mapping $w = \cos(z)$. Describe how the boundary and the interior of the region are mapped into their images. Hint: use the identities

$$\cos(z) = \sin\left(z + \tfrac{\pi}{2}\right) \quad \text{or} \quad \cos(x+iy) = \cos x \cosh y - i \sin x \sinh y.$$





For $-\pi < x < -\frac{\pi}{2}$, $y > 0$, $\cos x < 0$, $\sin x < 0$

$\Rightarrow \cos(x+iy) = -a\cosh y + bi\sinh y$

$= u + vi$

— hyperbolas with $u < 0$, $v > 0$

$x = -\frac{\pi}{2} \longmapsto u = 0,\ v \geq 0$ (pos. v-axis)

$x = -\pi \longmapsto v = 0,\ u \leq -1$ (interval $(-\infty, -1]$)

$\left\{ \begin{array}{l} -\pi < x < -\frac{\pi}{2} \\ y = 0 \end{array} \right\} \longmapsto \left\{ \begin{array}{l} -1 < u < 1 \\ v = 0 \end{array} \right\}$ (interval $[-1, 0]$)