March 15, 2019 MATH 655 Prof. V. Panferov

Practice Problems: Harmonic Functions

1. If $u(x,y)$ is harmonic, prove that

$$f(z) = u_x - iu_y$$

is analytic.

2. (a) Show that in polar coordinates (r,θ) Laplace's equation $u_{xx} + u_{yy} = 0$ takes the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

   (b) Conclude that the most general harmonic function that depends on r only is $a\ln r + b$ with a and b real constants.

3. Find the most general harmonic polynomial of the form

$$P(x,y) = ax^3 + bx^2y + cxy^2 + dy^3.$$

[Hint: the harmonic conjugate of P must be a polynomial of the same form (up to a constant). What is the most general form of an analytic function that has this form?]

4. Suppose we know how to find the solution of the Dirichlet problem:

$$\begin{cases} \Delta u = 0 & \text{in } U \\ u = \varphi & \text{on } \Gamma \end{cases}$$

for any φ. Describe how to use these solutions to find v satisfying the Neumann problem:

$$\begin{cases} \Delta v = 0 & \text{in } U \\ \frac{\partial v}{\partial n} = \psi & \text{on } \Gamma \end{cases}$$

for a given function ψ.

5. Suppose we know how to find all solutions of the Neumann problem:

$$\begin{cases} \Delta v = 0 & \text{in } U \\ \frac{\partial v}{\partial n} = \psi & \text{on } \Gamma \end{cases}$$

for any ψ. Describe how use these solutions to find u satisfying the Dirichlet problem:

$$\begin{cases} \Delta u = 0 & \text{in } U \\ u = \varphi & \text{on } \Gamma \end{cases}$$

for a given function φ.

6. (Problem 2-5:2) Use the Maximum Principle for harmonic functions (p. 46) to show that the solution of the Dirichlet problem is unique, and that the solution of any properly posed Neumann problem is unique within an arbitrary additive constant. *[Hint: Suppose that two harmonic functions u_1 and u_2 both satisfy the same Dirichlet boundary data $u_{1,2} = \varphi$ on Γ. Consider $u = u_1 - u_2$ which is also harmonic, and satisfies the condition $u = 0$ on Γ.]*

7. (Problem 2-5:3) Use Poisson's integral formula for the unit disk (p. 47) to obtain analogous formulas for the solutions of the Dirichlet and Neumann problems in a disk of arbitrary radius R centered at zero.

8. Show that the mean-value formula

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta})\, d\theta$$

remains valid for $f(z) = \ln|1+z|$, $z_0 = 0$, $r = 1$, and use this fact to compute

$$\int_0^{\pi} \ln \sin\theta \; d\theta.$$