Mon, Mar 4, 2019 **MATH 655, Midterm 1** Prof. V. Panferov

Name: (print) ______________________

CSUN ID No. : Solutions.

This test contains 8 questions, on 8 pages. The perfects score is 42 points, the last question is a bonus worth an extra 6 points. The duration of the test is 1 hour 15 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	8	total

Important: The test is closed books/notes. No electronic devices; all cellphones must be turned off and put away for the duration of the test. Show all your work.

1. (6 points) Give an example of a power series whose radius of convergence is 2, and which converges uniformly in the disk $|z| < 2$. Justify your answer.

Example: $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 2^n} z^n$$

By the ratio test,

$$\frac{a_{n+1}}{a_n} = \left| \frac{(-1)^{n+2}\, n^2 2^n}{(-1)^{n+1}(n+1)^2\, 2^{n+2}} \right| = \left(\frac{n}{n+1}\right)^2 \frac{1}{2} \longrightarrow \frac{1}{2}$$

$\Rightarrow$ the series has radius of convergence 2.

Also when $|z| \le 2$,

$$\left| \frac{(-1)^{n+1}}{n^2 2^n} z^n \right| \le \frac{1}{n^2}$$ which forms a convergent series.

By Weierstrass test the series converges uniformly for $|z| \le 2$ and therefore for $|z| < 2$.

2. (6 points) Let

$$f(z) = \lim_{n\to\infty} \frac{z^n}{n+z^n}.$$

(a) Find an explicit formula for $f(z)$.

$$|z|<1 \Rightarrow |z^n| = |z|^n \leq 1$$

$$\Rightarrow \left|\frac{z^n}{n+z^n}\right| \leq \frac{|z|^n}{n-|z|^n} = \frac{1}{n}\,\frac{|z|^n}{1-\frac{|z|^n}{n}} \leq \frac{1}{n}\,\frac{1}{1-\frac{1}{n}} \to 0$$

$$|z|>1 \Rightarrow \frac{z^n}{n+z^n} = \frac{1}{1+nz^{-n}} \to 1$$

since $|nz^{-n}| = n|z|^{-n} \to 0$

$$\Rightarrow f(z) = \begin{cases} 0, & |z|\leq 1 \\ 1, & |z|>1. \end{cases}$$

(b) Is the convergence of the functions $\dfrac{z^n}{n+z^n}$ uniform on $|z|<1$? On $|z|>1$? Justify your answer.

Since

$$\left|\frac{z^n}{n+z^n}\right| \leq \frac{1}{n}\,\frac{1}{1-\frac{1}{n}} \to 0 \quad \text{for } |z|\leq 1$$

the sequence converges uniformly on $|z|\leq 1$ $(\Rightarrow |z|<1)$

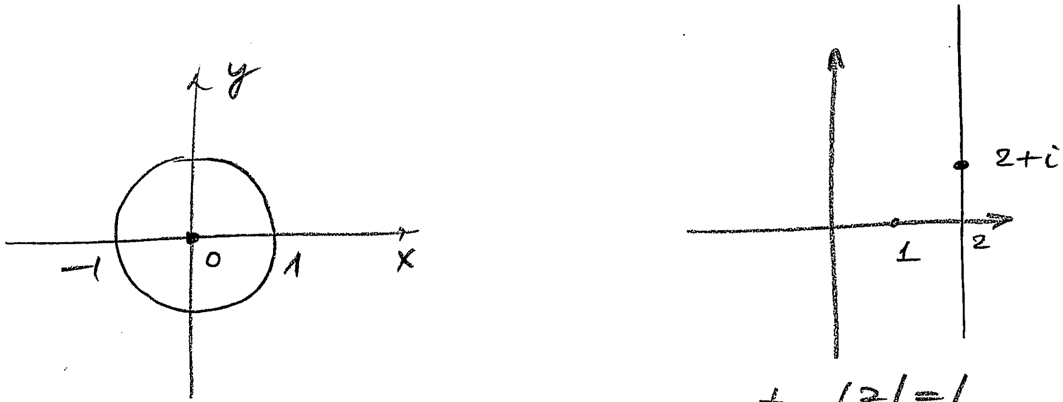
For $|z|>1$

$\left|\dfrac{z^n}{n+z^n} - 1\right|$ is unbounded, since

$|n+z^n| < \delta$ for z close enough to $(-n)^{\frac{1}{n}}$

so no uniform convergence can be expected.

3. (6 points) Find the fractional linear transformation which carries 0 into 1, 1 into $2+i$, and the circle $|z|=1$ into the line $\mathrm{Re}(w)=2$.



∞ is symmetric to 0 w.r.t $|z|=1$

3 is symmetric to 1 w.r.t $\mathrm{Re}(w)=2$.

$$(z, 0, 1, \infty) = (w, 1, 2+i, 3)$$

$$\frac{z-1}{z-\infty} : \frac{0-1}{0-\infty} = \frac{w-(2+i)}{w-3} : \frac{1-(2+i)}{1-3}$$

work it out, or since $\infty \to 3$

$$w = \frac{3z+b}{z+d}$$

since $0 \to 1 \Rightarrow \frac{b}{d} = 1$

$$w = \frac{3z+b}{z+b}$$

Since $1 \to 2+i$

$$\frac{3+b}{1+b} = 2+i$$

$$b = \frac{(2+i)-3}{1-(2+i)} = \frac{-1+i}{-1-i} = \frac{1-i}{1+i} = -i$$

$$w = \frac{3z-i}{z-i}.$$

Continued...

4. (6 points) If $f(z) = z\text{Re}(z)$ prove that f is not differentiable except at $z = 0$. Find $f'(0)$.

$$f'(z) = \lim_{\xi \to z} \frac{\xi\, Re(\xi) - z\, Re(z)}{\xi - z}$$

$$= \lim_{\xi \to z} \frac{(\xi - z)\, Re(\xi) + z\,(Re(\xi) - Re(z))}{\xi - z}$$

$$= \lim_{\xi \to z} Re(\xi) + z\, \frac{Re(\xi - z)}{\xi - z}$$

If $\xi - z = re^{i\theta}$ then $Re(\xi - z) = r\cos\theta$

and $\dfrac{Re(\xi - z)}{\xi - z} = \dfrac{\cos\theta}{\cos\theta + i\sin\theta}$

$$= \cos^2\theta - i\cos\theta\sin\theta$$

Since the limit depends on the mode of approach, the $\lim \xi \to z$ does not exist, unless $z = 0$.

In that case

$$f'(0) = \lim_{\xi \to 0} Re(\xi) = 0.$$

5. (6 points) Find the integrals:

(a) $\int_C e^{iz}\, dz$, from the origin to the point $\frac{\pi}{2}+i$ taken along the parabola $y = \left(\frac{2}{\pi}\right)^2 x^2$.

e^{iz} is analytic on $\mathbb{C}$;
integral is path-independent

$$\int_0^{\frac{\pi}{2}+i} e^{iz}\, dz = \left[\frac{e^{iz}}{i}\right]_0^{\pi/2+i}$$

$$= \frac{e^{-1+\frac{\pi}{2}i} - 1}{i} = \frac{ie^{-1}-1}{i}$$

$$= e^{-1}+i.$$

(b) $\int_C \frac{z+1}{z^2+z+1}\, dz$ along the circle $|z| = \frac{1}{2}$, in the counterclockwise direction.

z^2+z+1 has zeros $z = -\frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 - 1}$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$z^2+z+1 \neq 0$ for $|z| < 1$

$\Rightarrow \frac{z+1}{z^2+z+1}$ is analytic in $|z|<1$

By Cauchy's Theorem,

$$\int_C \frac{z+1}{z^2+z+1}\, dz = 0.$$

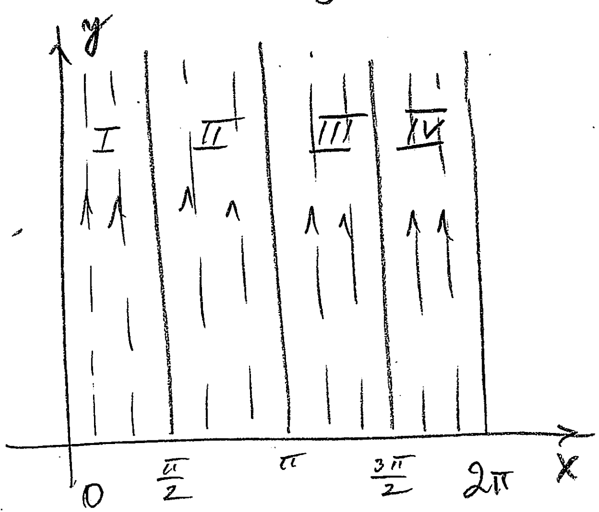
Continued...

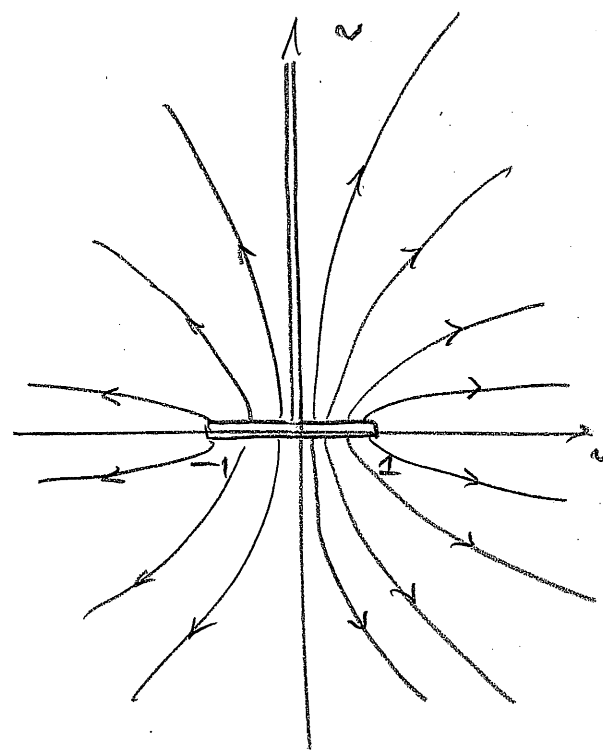
6. (6 points) Describe graphically the action of the transformation $w = \sin(z)$ on the region

$$R = \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 2\pi,\ \mathrm{Im}(z) > 0\}.$$

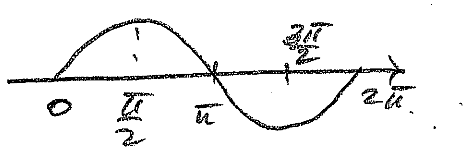
Show the images of the coordinate lines $\{\mathrm{Re}(z) = \text{const}\}$ and $\{\mathrm{Im}(z) = \text{const}\}$ in the w-plane. What is the image of R? What happens to the boundary of the region?

$\sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$

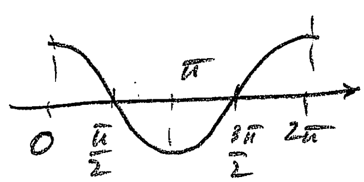




sin(x)



cos(x)



I: $\sin(x) > 0$, $\cos(x) > 0$ $\Rightarrow$ (u,v) in 1st quadr

$x = \text{const}$ – hyperbolas

$y = \text{const}$ ellipses

II $\sin(x) > 0$, $\cos(x) < 0$ $\Rightarrow$ (u,v) in 4th quadr

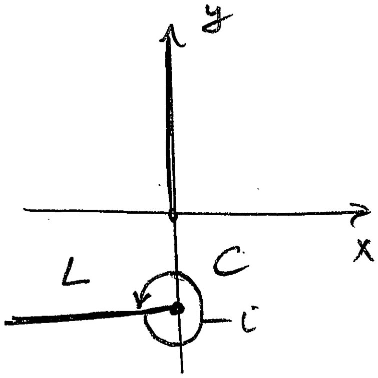
III $\sin(x) < 0$, $\cos(x) < 0$ $\Rightarrow$ (u,v) in 3rd quadr.

IV $\sin(x) < 0$, $\cos(x) > 0$ $\Rightarrow$ (u,v) in 2nd quadr.

$[0, 2\pi] \to [-1, 1]$ (twice); $\left.\begin{matrix} x = 0 \\ x = 2\pi \end{matrix}\right\} \to u = 0,\ v > 0$ (once each)

Continued...

7. (6 points) (a) Let $R = \mathbb{C} \setminus i[0, \infty)$. Prove that it is impossible to define a continuous branch of the function $f(z) = z + \sqrt{1+z^2}$ on R.



z – continuous everywhere

$\sqrt{1+z^2}$ has branch points at $\pm i$

$z=-i$ is not covered by the cut.

Let $\sqrt{1+z^2} = \sqrt{\underbrace{(z-i)}_{\rho_1 e^{i\vartheta_1}}\underbrace{(z+i)}_{\rho_2 e^{i\vartheta_2}}} = (\rho_1\rho_2)^{\frac{1}{2}} e^{i\frac{1}{2}(\vartheta_1+\vartheta_2)} e^{i\pi n}$ $(n=0,1)$

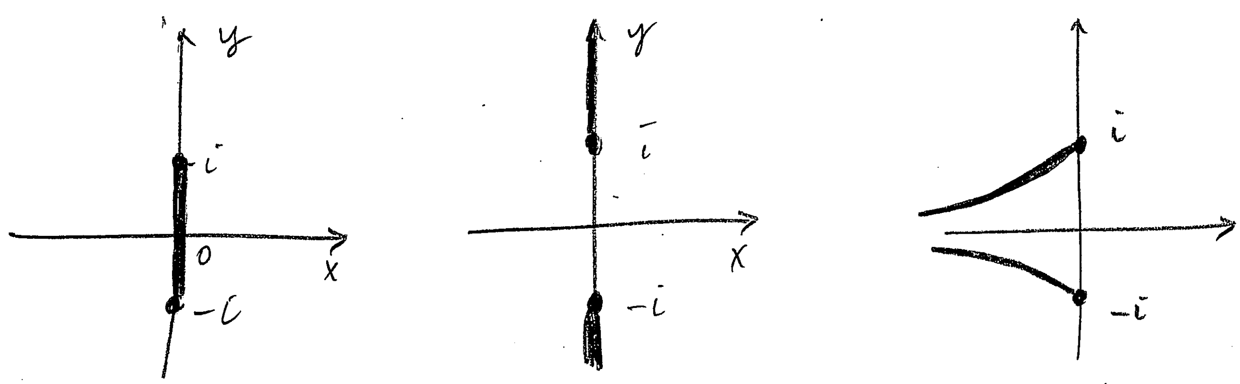
Take C – circle centered at $-i$ of small radius.

Then $\Delta_C \vartheta_1 = 0$, $\Delta_C \vartheta_2 = 2\pi$

$$\Delta_C e^{i(\vartheta_1+\vartheta_2)/2} = e^{i\pi} = -1$$

With an arbitrary cut through $z=-i$, say $L = (-\infty, 0] - i$, take $z_0 \in L$. Fixing any of the branches of $\sqrt{1+z^2}$, the limits on the opposite sides of the cut are $z_0 + \sqrt{1+z_0^2}$ and $z_0 - \sqrt{1+z_0^2}$ – different $\Rightarrow f(z)$ is not continuous at z_0.

(b) Describe possible branch cuts in the complex plane such that $f(z) = z + \sqrt{1+z^2}$ has a continuous branch in $\mathbb{C} \setminus \{\text{cuts}\}$. Justify your answer.



etc.

Any collection of curves such that encircling i or $-i$ is impossible in $\mathbb{C} \setminus \{\text{cuts}\}$.

$$f(z) = (\rho_1\rho_2)^{\frac{1}{2}} e^{i\frac{1}{2}(\vartheta_1+\vartheta_2)} e^{i\pi n}, \quad n=0,1$$

are 2 continuous branches for any of the possible cuts above.

8. (a) (bonus: 2 points) Describe possible branch cuts in the complex plane such that $f(z) = \ln(z + \sqrt{1+z^2})$ has a continuous branch in $\mathbb{C} \setminus \{\text{cuts}\}$. Justify your answer.

The 1st choice in 7(a) does not work since ∞ is a branch point of ln.

WLOG let $\sqrt{1+z^2}$ denote the branch so that $\sqrt{1+z^2}\Big|_{z=0} = 1$

otherwise use $\ln(z - \sqrt{1+z^2}) = -\ln(z + \sqrt{1+z^2})$.

Then $\ln(z+\sqrt{1+z^2}) = \ln z\left(1+\sqrt{1+\frac{1}{z^2}}\right) = \ln z + \ln\left(1+\sqrt{1+\frac{1}{z^2}}\right)$

On a circle of large radius, (C) $\ln\left(1+\sqrt{1+\frac{1}{z^2}}\right) \approx \ln(2)$

Therefore $\Delta_C \ln(z+\sqrt{1+z^2}) = \Delta_C \ln(z) = 2\pi i$

By the same argument as in 7(a) $f(z)$ cannot be continuous on $\mathbb{C} - i[-1,1]$. The other choices (cuts extending to ∞) are OK.

(b) (bonus: 4 points) If $\lim\limits_{z\to z_0} \left|\dfrac{f(z)-f(z_0)}{z-z_0}\right|$ exists, and $f = \underbrace{u+iv}_{\text{differentiable w.r.t. } x,y}$ show that either f or f^* must be differentiable at z_0.

u, v - differentiable $\Rightarrow$

$$\lim_{r\to 0} \frac{f(z_0+re^{i\theta}) - f(z_0)}{re^{i\theta}} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial z^*} e^{-2i\theta}$$

Then

$$\lim_{r\to 0} \left|\frac{f(z_0+re^{i\theta}) - f(z_0)}{re^{i\theta}}\right| = \left|\frac{\partial f}{\partial z} + \frac{\partial f}{\partial z^*} e^{-2i\theta}\right|$$

The limit is θ-independent $\Leftrightarrow$ $\frac{\partial f}{\partial z} = 0$ or $\frac{\partial f}{\partial z^*} = 0$

In the first case $f^{*\prime}(z) = \left(\frac{\partial f}{\partial z^*}\right)^*$

In the second case $f'(z) = \frac{\partial f}{\partial z}$.