HOMEWORK 9 – SOLUTIONS

1.

Expand $\frac{1}{\sin z}$ in powers of z for

(a) $0 < |z| < \pi$

and (b) for $\pi < |z| < 2\pi$.

(a): From Problem 5(c) in Hw Set 8,

$$\frac{1}{\sin z} = \frac{1}{2}\cot\left(\frac{z}{2}\right) - \frac{1}{2}\cot(z) = \frac{2}{z}\sum_{n=0}^{\infty} \frac{(-1)^n B_{2n}(1-2^{2n-1})}{(2n)!} z^{2n}$$

$$= \frac{1}{z} + \frac{1}{6}z + \frac{7}{360}z^3 + \dots = \sum_{k=-\infty}^{\infty} a_k z^k.$$

(b) Let $\frac{1}{\sin z} = \sum_{k=-\infty}^{\infty} b_k z^k$, then

$$b_k = \frac{1}{2\pi i}\int_{C_1} \frac{1}{\sin z} z^{-(k+1)}\, dz$$

where C_1 is any contour on $\pi < |z| < 2\pi$

$$b_k - a_k = \frac{1}{2\pi i}\left(\int_{C_1} - \int_{C_0}\right)\frac{1}{\sin z} z^{-(k+1)}\, dz$$

$$= \sum_{z=\pm\pi} \operatorname{Res} \frac{1}{\sin z} z^{-(k+1)}$$

$$= \frac{1}{\pi^{k+1}} \operatorname{Res}_{z=\pi} \frac{1}{\sin z} + (-1)^{k+1}\frac{1}{\pi^{k+1}} \operatorname{Res}_{z=-\pi} \frac{1}{\sin z}$$

$$= \frac{1}{\pi^{k+1}}\left((-1)^k - 1\right) \neq 0 \text{ if } k \text{ is odd}$$

$$\Rightarrow b_k = \frac{1}{\pi^{k+1}}\left((-1)^k - 1\right) + a_k.$$

#2. $f(z) = \exp(t(z+z^{-1}))$

Laurent series: $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n; \quad z_0 = 0$

$$a_n = \frac{1}{2\pi i}\int_C f(\xi)\,\xi^{-(n+1)}\,d\xi$$

Choose $C = \{|z|=1\}$; $\xi = e^{i\vartheta}$; $d\xi = i\xi\, d\vartheta$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\vartheta})\, e^{-ni\vartheta}\, d\vartheta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{t(e^{i\vartheta}+e^{-i\vartheta})}\, e^{-ni\vartheta}\, d\vartheta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{2t\cos\vartheta}\left(\cos(n\vartheta) - i\sin(n\vartheta)\right) d\vartheta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{2t\cos\vartheta}\cos(n\vartheta)\, d\vartheta$$

(The imaginary part vanishes, since the integrand is odd about $\vartheta = \pi$.)

Also: $f(z) = e^{tz}\cdot e^{tz^{-1}} = \sum_{n=0}^{\infty} \frac{t^n z^n}{n!} \sum_{m=0}^{\infty} \frac{t^m z^{-m}}{m!}$

$$= \sum_{k=-\infty}^{\infty} z^k \sum_{n=0}^{\infty} \frac{t^{2n+|k|}}{n!\,(n+|k|)!} = \sum_{k=-\infty}^{\infty} z^k I_{|k|}(2t),$$

where

$$I_k(t) = \sum_{n=0}^{\infty} \frac{1}{n!\,(n+k)!}\left(\frac{t}{2}\right)^{2n+k}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{t\cos\vartheta}\cos(k\vartheta)\, d\vartheta.$$

– modified Bessel function of the first kind.

#3. $f(z) = \sqrt{z}\,(1+\sin z)^{-1}$

Singularities: branch point at $z=0$

poles of order 2 at $z = -\frac{\pi}{2} + 2\pi n$

(since $1 + \sin z = 1 - \cos\left(z + \frac{\pi}{2}\right) = \sin^2 \frac{z+\frac{\pi}{2}}{2}$

zeros of $\sin w$ are $w = \pi n,\ n \in \mathbb{Z}$)

$z = \infty$ is a branch point,

since encircling ∞ along a contour avoiding the poles switches branches of $\sqrt{z}$.

$z=\infty$ is also a cluster point for the sequence of poles $z_n = -\frac{\pi}{2} + 2\pi n$.

At $z = -\frac{\pi}{2}$:

$$\sqrt{z} = \sqrt{-\frac{\pi}{2} + \left(z+\frac{\pi}{2}\right)} = \pm i\sqrt{\frac{\pi}{2}}\sqrt{1 - \frac{z+\frac{\pi}{2}}{\pi/2}}$$

$$= \pm\sqrt{\frac{\pi}{2}}\, i \left(1 - \frac{1}{2}\left(\frac{z+\frac{\pi}{2}}{\pi/2}\right) + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}\left(\frac{z+\pi/2}{\pi/2}\right)^2 - \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(\frac{z+\pi/2}{\pi/2}\right)^3 + \cdots\right)$$

$$= \pm\sqrt{\frac{\pi}{2}}\, i \left(1 - \frac{1}{\pi}\left(z+\frac{\pi}{2}\right)^2 - \frac{1}{2\pi^2}\left(z+\frac{\pi}{2}\right)^2 - \frac{1}{2\pi^3}\left(z+\frac{\pi}{2}\right)^3 + \cdots\right)$$

$$\frac{\sqrt{z}}{1+\sin z} = \frac{\sqrt{z}}{\sin^2\left(\frac{z+\pi/2}{2}\right)} = \frac{\pm\sqrt{\frac{\pi}{2}}\, i\left(1 - \frac{1}{\pi}\left(z+\frac{\pi}{2}\right)^2 - \cdots\right)}{\left(\frac{z+\pi/2}{2} - \frac{1}{6}\left(\frac{z+\pi/2}{2}\right)^3 + \cdots\right)^2}$$

$$= \pm\sqrt{\frac{\pi}{2}}\, i\left(\frac{2}{z+\pi/2}\right)^2 \frac{\left(1 - \frac{1}{\pi}\left(z+\frac{\pi}{2}\right)^2 - \frac{1}{2\pi^2}\left(z+\frac{\pi}{2}\right)^3 - \cdots\right)}{\left(1 - \frac{1}{3}\left(\frac{z+\pi/2}{2}\right)^2 + \cdots\right)}$$

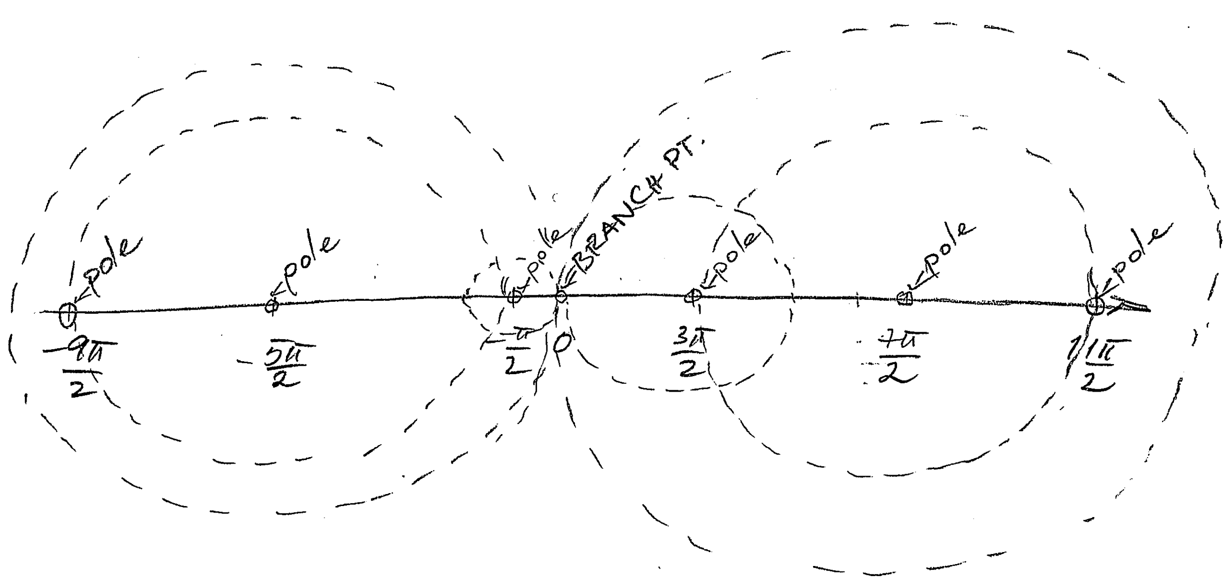
$$= \pm\sqrt{\frac{\pi}{2}}\, i\left(\frac{2}{z+\pi/2}\right)^2\left(1 - \left(\frac{1}{\pi} - \frac{1}{12}\right)\left(z+\frac{\pi}{2}\right)^2 + \cdots\right)$$

The series converges on the region $0<\left|z+\frac{\pi}{2}\right|<\frac{\pi}{2}$

Laurent series on other annular regions about $z=-\frac{\pi}{2}$ cannot be constructed because of the branch singularity of $f(z)$ at zero.

Laurent series on other rings centered on $z_n=-\frac{\pi}{2}+2\pi n$, $n\in\mathbb{Z}$ can be constructed as long as $z=0$ is outside the region.

Laurent series are impossible to construct for any n-d of infinity because of the accumulating poles and branch point at ∞.



Different possibilities for regions with Laurent series expansions.

#4. A limit point of poles (z_0) is not an isolated singularity (same applies to a finite z_0 and $z_0 = \infty$). Therefore, Laurent series cannot be constructed (the sum of a Laurent series is necessarily analytic on a ring) and the point z_0 cannot be a pole.

Also for any $w \in \mathbb{C}$, $\dfrac{1}{f(z)-w}$ is un-bounded as $z \to z_0$, for if it is bounded, the function $g(z) = \dfrac{1}{f(z)-w}$ has zeros on a sequence $z_k \to z_0$ and is bounded as $z \to z_0 \Rightarrow z_0$ is a removable singularity of $g(z) \Rightarrow g(z) \equiv 0$ on a n-d of z_0, which is impossible

For instance, $\csc \frac{1}{z}$ assumes values arb. close to any prescribed $w \in \mathbb{C}$ as $z \to 0$, $z \neq \pi n$.

In this way, $z=0$ works similar to an essential singularity for the function $\csc \frac{1}{z}$.