HOMEWORK 8 — SOLUTIONS

#1: $f(z)$ analytic for $0 < |z-z_0| < \rho$; $|f(z)|$ bounded

Consider $h(z) = f(z)(z-z_0)^2$

Then $$h'(z_0) = \lim_{z \to z_0} \frac{f(z)(z-z_0)^2}{z-z_0} = 0$$

since $$\left| \frac{f(z)(z-z_0)^2}{z-z_0} \right| \le C|z-z_0| \underset{z \to z_0}{\longrightarrow} 0.$$

Therefore $h(z)$ is analytic for $|z-z_0| < \rho$

$\Rightarrow$ $h(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$

where $a_0 = h(z_0) = 0$, $a_1 = h'(z_0) = 0$

$\Rightarrow$ $f(z) = a_2 + a_3(z-z_0) + a_4(z-z_0)^2 + \cdots$

for $0 < |z-z_0| < \rho$.

Re-define $f(z_0) = a_2$, then the new function $f(z)$ is represented by a convergent series for $|z-z_0| < \rho$

$\Rightarrow$ analytic on the disk $|z-z_0| < \rho$.

#2. Consider $g(z) = \frac{z-1}{z+1}$ — maps the right half-plane onto the unit circle.

Then $h(z) = g(f(z))$ maps $0 < |z - z_0| < \rho$ into $|w| \le 1$

so $|h(z)| \le 1 \Rightarrow h(z)$ is analytic for $|z - z_0| < \rho$ by Problem 1.

Then $f(z) = \dfrac{h(z)+1}{1-h(z)}$ is analytic for $|z-z_0| < \rho$ $(h(z) \neq 1)$

if we adjust the value $f(z_0)$ in such a way that $h(z_0)$ is analytic at z_0.

#3. By Problem 6 on Homework Set 7,

$$\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,d\theta = \alpha \ln r + \beta$$

If u is bounded, then the integral remains bounded as $r \to 0 \Rightarrow \alpha = 0$.

Therefore $\alpha = \int_{|z|=r} \frac{\partial u}{\partial n}\,ds = \int_{|z|=r} -\frac{\partial v}{\partial s}\,ds = 0$,

and same is true for any closed simple contour C in $|z|<\rho$.

(C is either homologous to 0 or homologous to $|z|=r$ in $U = \{|z|<\rho\}$.)

$$\Rightarrow v = v(x_1, y_1) + \int_{(x_1,y_1)}^{(x,y)} -u_y\,dx + u_x\,dy$$

is path-independent in $0 < |z| \ldots$

Then $f(z) = u + iv$ is analytic and single-valued for $0 < |z - z_0| < \rho$ and $\mathrm{Re}(f(z))$ is bounded. By Problem 2, can redefine $f(0)$ so $f(z)$ is analytic for $|z|<\rho$.

#4 (a) $f(z) = (1+z+z^2)^{-1}$; $z_0 = 0$

$$1+z+z^2 = (z-z_1)(z-z_1^*)$$

$$1+z+z^2 = \left(z+\frac{1}{2}\right)^2 + \frac{3}{4} \; ; \quad z_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2} = e^{i2\pi/3}.$$

$$\frac{1}{1+z+z^2} = \left(\frac{1}{z-z_1} - \frac{1}{z-z_1^*}\right)\frac{1}{z_1 - z_1^*}$$

$$= \frac{1}{2i\,\mathrm{Im}(z_1)}\left(\frac{1}{z-z_1} - \frac{1}{z-z_1^*}\right).$$

$$\frac{1}{z-z_1} = \frac{1}{-z_1\left(1-\frac{z}{z_1}\right)} = -\frac{1}{z_1}\sum_{k=0}^{\infty}\left(\frac{z}{z_1}\right)^k$$

$$= -\sum_{k=0}^{\infty}\frac{z^k}{z_1^{k+1}}$$

$$\frac{1}{z-z_1^*} = -\sum_{k=0}^{\infty}\frac{z^k}{z_1^{*\,k+1}}$$

$$\frac{1}{1+z+z^2} = \sum_{k=0}^{\infty}\frac{1}{2i\,\mathrm{Im}(z_1)}\left(\frac{1}{z_1^{*\,k+1}} - \frac{1}{z_1^{k+1}}\right)z^k$$

$$= \sum_{k=0}^{\infty}\frac{1}{2i\sin\left(\frac{2\pi}{3}\right)}\left(e^{i(k+1)2\pi/3} - e^{i(k+1)2\pi/3}\right)z^k$$

$$= \sum_{k=0}^{\infty}\frac{\sin\left((k+1)\frac{2\pi}{3}\right)}{\sin\frac{2\pi}{3}}z^k$$

$$= 1 - z + z^3 - z^4 + z^6 - z^7 + \cdots$$

(b) $\sin^2 z = \dfrac{1-\cos(2z)}{2}$; $z_0 = 0$, $z_0 = -1$

$\underline{z_0 = 0}$: $\cos z = 1 - \dfrac{z^2}{2} + \dfrac{z^4}{24} - \dots = \sum_{k=0}^{\infty} \dfrac{(-1)^k}{(2k)!} z^{2k}$

$$\cos(2z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} 2^{2k} z^{2k}$$

$$\sin^2 z = \frac{1}{2} - \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} 2^{2k-1} z^{2k}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k)!} 2^{2k-1} z^{2k}$$

$$= z^2 - \frac{z^4}{3} + \frac{2}{45} z^6 - \dots$$

$\underline{z_0 = -1}$: $\cos(2(z+1)-2) = \cos(2(z+1))\cos(2) + \sin(2(z+1))\sin(2)$.

$$\cos(2(z+1)) = 1 - 2(z+1)^2 + \frac{2}{3}(z+1)^4 - \dots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} 2^{2k} (z+1)^{2k}$$

$$\sin(2(z+1)) = 2(z+1) - \frac{4}{3}(z+1)^3 + \dots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} 2^{2k+1} z^{2k+1}$$

$$\sin^2 z = \frac{1}{2} - \frac{1}{2}\cos(2(z+1)-2)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!} 2^{2k} \sin(2)\, z^{2k+1}$$

$$\frac{1}{2} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k)!} 2^{2k-1} \cos(2)\, z^{2k}$$

$$= \left(\frac{1}{2} - \frac{\cos(2)}{2}\right) - \sin(2)(z+1) + \cos(2)(z+1)^2 + \frac{2}{3}\sin(2)(z+1)^3 - \frac{1}{3}\cos(2)(z+1)^4 - \dots$$

$$= \sin(2) - \sin(2)(z+1) + \cos(2)(z+1)^2 + \frac{2}{3}\sin(2)(z+1)^3 - \frac{1}{3}\cos(2)(z+1)^4 - \dots$$

(c) $f(z) = z^{1/2}$; $z_0 = 1$, $z_0 = i\pi$.

$\underline{z_0 = 1}$:

$$z^{\frac{1}{2}} = \left(1 + (z-1)\right)^{\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{1/2}{k} (z-1)^k$$

$$= 1 + \frac{1}{2}(z-1) + \frac{1/2 \cdot (-1/2)}{2!}(z-1)^2 + \frac{1/2(-1/2)(-3/2)}{3!}(z-1)^3 + \cdots$$

$$= 1 + \frac{1}{2}(z-1) - \frac{1}{8}(z-1)^2 + \frac{1}{16}(z-1)^3 + \cdots$$

– for the principal branch;

$$z^{\frac{1}{2}} = -\left(1 + \frac{1}{2}(z-1) - \frac{1}{8}(z-1)^2 + \frac{1}{16}(z-1)^3 - \cdots\right)$$

for the 2nd branch.

$\underline{z_0 = i\pi}$:

$$z^{\frac{1}{2}} = \left(i\pi + (z - i\pi)\right)^{\frac{1}{2}} = (i\pi)^{\frac{1}{2}}\left(1 + \frac{z - i\pi}{i\pi}\right)^{\frac{1}{2}}$$

$$= (i\pi)^{\frac{1}{2}} \sum_{k=0}^{\infty} \binom{1/2}{k} \frac{(z-i\pi)^k}{(i\pi)^k}$$

$$= (i\pi)^{\frac{1}{2}}\left(1 + \frac{1}{2}\,\frac{z-i\pi}{i\pi} - \frac{1}{8}\,\frac{(z-i\pi)^2}{(i\pi)^2} + \frac{1}{16}\,\frac{(z-i\pi)^3}{(i\pi)^3} - \cdots\right)$$

$$(i\pi)^{\frac{1}{2}} = \pm\sqrt{\pi}\, e^{i\pi/4} = \pm \frac{\sqrt{\pi}}{\sqrt{2}}(1+i)$$

for the two branches of the square root fn.

(d) $\quad f(z) = \ln\left(iz + (1-z^2)^{1/2}\right) \quad ; \quad z_0 = 0; \quad z_0 = i$ $\quad$ 7

$$\ln\left(iz + (1-z^2)^{1/2}\right)' = \frac{i + \frac{-z}{\sqrt{1-z^2}}}{iz + \sqrt{1-z^2}} = \frac{i}{\sqrt{1-z^2}}$$

(two branches differ by "$\pm$")

$\underline{z_0 = 0}$:

$$\frac{i}{\sqrt{1-z^2}} = i\,(1-z^2)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} i \binom{-1/2}{k} (-z^2)^k$$

$$= \sum_{k=0}^{\infty} (-1)^k i \binom{-1/2}{k} z^{2k}.$$

$$\ln\left(iz + (1-z^2)^{1/2}\right) = 2\pi i n \pm \sum_{k=0}^{\infty} \frac{(-1)^k i \binom{-1/2}{k}}{2k+1} z^{2k+1}$$

$\underline{z_0 = i}$: $\quad$ set $z = i + w$, $\quad w = z - i$

$$\frac{i}{\sqrt{1-z^2}} = \frac{i}{\sqrt{1-(i+w)^2}} = \frac{i}{\sqrt{2 - 2iw - w^2}}$$

$$= \frac{i}{\sqrt{2}} \left(1 - iw - \frac{w^2}{2}\right)^{-1/2}$$

$$= \frac{i}{\sqrt{2}} \left(1 - \frac{1}{2}\left(-iw - \frac{w^2}{2}\right) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}\left(-iw - \frac{w^2}{2}\right)^2 \right.$$

$$\left. + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{6}\left(-iw - \frac{w^2}{2}\right)^3 + \ldots \right)$$

$$= \frac{i}{\sqrt{2}} \left(1 + \frac{i}{2}w + \left(\frac{1}{4} - \frac{3}{8}\right)w^2 + \left(\frac{3}{8} - \frac{5}{16}\right) i w^3 + \ldots\right)$$

$$= \frac{i}{\sqrt{2}} \left(1 + \frac{i}{2}w - \frac{1}{8}w^2 + \frac{1}{16} i w^3 + \ldots\right)$$

$$\ln\left(iz+(1-z^2)^{1/2}\right) = 2\pi i n \pm \frac{i}{\sqrt{2}}\left(w + \frac{i}{4}w^2 - \frac{1}{24}w^3 + \frac{1}{64}iw^4 + \cdots\right)$$

$$= 2\pi i n \pm \frac{i}{\sqrt{2}}\left((z-i) + \frac{i}{4}(z-i)^2 - \frac{1}{24}(z-i)^3 + \frac{1}{64}i(z-i)^4 + \cdots\right).$$

#5 (a)

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = 1 - \frac{1}{2}z + \frac{z^2}{12} - \frac{1}{720}z^4 + \cdots$$

$$\frac{z}{2} + \frac{z}{e^z-1} = \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} z^{2n}.$$

$$\cot z = \frac{\cos z}{\sin z} = i\,\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}} = i\,\frac{e^{2iz}+1}{e^{2iz}-1}$$

$$= i\left(\frac{e^{2iz}-1}{e^{2iz}-1} + \frac{2}{e^{2iz}-1}\right) = \frac{1}{z}\left(\frac{2iz}{2} + \frac{2iz}{e^{2iz}-1}\right)$$

$$= \frac{1}{z}\sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!}(2iz)^{2n} = \frac{1}{z}\sum_{n=0}^{\infty} \frac{(-1)^n B_{2n}\, 2^{2n}}{(2n)!} z^{2n}.$$

(b)

$$\tanh z + \coth z = \frac{e^z-e^{-z}}{e^z+e^{-z}} + \frac{e^z+e^{-z}}{e^z-e^{-z}}$$

$$= \frac{(e^z-e^{-z})^2 + (e^z+e^{-z})^2}{e^{2z}-e^{-2z}} = \frac{2(e^{2z}+e^{-2z})}{e^{2z}-e^{-2z}} = 2\coth(2z)$$

$$\Rightarrow \tanh(z) = 2\coth(2z) - \coth z$$

$$= \frac{2}{2z}\sum_{n=0}^{\infty} \frac{B_{2n}\, 2^{2n}}{(2n)!} 2^{2n}\cdot z^{2n} - \frac{1}{z}\sum_{n=0}^{\infty} \frac{B_{2n}\, 2^{2n}}{(2n)!} z^{2n}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty} \frac{B_{2n}\, 2^{2n}(2^{2n}-1)}{(2n)!} z^{2n}$$

$$= \sum_{n=1}^{\infty} \frac{B_{2n}\, 2^{2n}(2^n-1)}{(2n)!} z^{2n-1}$$

since the $n=0$ term is zero.

(c)

$$\coth z = \frac{e^z+e^{-z}}{e^z-e^{-z}} = \frac{(e^z+e^{-z})(e^z+e^{-z})}{(e^z-e^{-z})(e^z+e^{-z})} = \frac{e^{2z}+e^{-2z}+2}{e^{2z}-e^{-2z}}$$

$$= \coth(2z) + 2\,\mathrm{csch}(2z)$$

$$\cot z = \cot(2z) + 2\csc(2z)$$

$$\csc(z) = \frac{1}{2}\cot\left(\frac{z}{2}\right) - \frac{1}{2}\cot(z)$$

$$= \frac{1}{z}\sum_{n=0}^{\infty} \frac{(-1)^n B_{2n} 2^{2n}}{(2n)!}\left(\frac{z}{2}\right)^{2n} - \frac{1}{2z}\sum_{n=0}^{\infty}\frac{(-1)^n B_{2n} 2^{2n}}{(2n)!} z^{2n}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n B_{2n}(1-2^{2n-1})}{(2n)!} z^{2n}$$

$$\frac{z-\pi}{\sin z} = -\frac{z-\pi}{\sin(z-\pi)} = -(z-\pi)\frac{1}{z-\pi}\sum_{n=0}^{\infty}\frac{(-1)^n B_{2n}(1-2^{2n-1})}{(2n)!} z^{2n}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n B_{2n}(2^{2n-1}-1)}{(2n)!} z^{2n}.$$

#6. Let $h(z) = \frac{f(z)}{z}$.

Since $f(z) = a_1 z + a_2 z^2 + \ldots$ then $h(z)$ is analytic for $|z|<1$. and $|h(z)| \le \frac{1}{|z|}$, $|z| \le 1$

By the Maximum Modulus Principle

$\forall a<1$ $|h(z)| \le \frac{1}{a}$, $|z| \le a$; since a is arbitrary $|h(z)| \le 1$, $|z| \le 1$.

Therefore $|f(z)| \le |z|$, $|z| \le 1$.

If the equality $|f(z)| = |z| \Leftrightarrow |h(z)| = 1$ is achieved for $z = z_0 \neq 0$ on $|z|<1$ then $h(z)$ achieves its maximum inside $|z|=1$

$\Rightarrow$ $h(z)$ is constant for $|z|<1$.

$\Rightarrow |h(z)| = 1$ for $|z|<1$

$\Rightarrow h(z) = e^{i\alpha}$, α - real

$\Rightarrow f(z) = e^{i\alpha} z$, $|z|<1$.

#7 (a) $f(z) = \sum_{n=0}^{\infty} a_n z^n$

$$a_n = \frac{1}{2\pi i} \int_C f(\xi)\, \xi^{-(n+1)}\, d\xi$$

$$\left(C = \{z : |z| = r\}\right) = \frac{1}{2\pi r^n} \int_0^{2\pi} f(re^{i\theta})\, e^{-in\theta}\, d\theta$$

$$= \frac{1}{2\pi r^n}\Big(\int_0^{\pi} f(re^{i\theta})\, e^{-in\theta}\, d\theta + \int_0^{\pi} f(re^{i(\theta+\pi)})\, e^{-in(\theta+\pi)}\, d\theta \Big)$$

$$= \frac{1}{2\pi r^n}\Big(\int_0^{\pi} f(re^{i\theta})\, e^{-in\theta}\, d\theta + e^{-in\pi} \int_0^{\pi} f(-re^{i\theta})\, e^{-in\theta}\, d\theta \Big)$$

if $f(z) = f(-z)$ and n is odd

Then $e^{-i\pi n} = -1 \Rightarrow a_n = 0$

if $f(z) = -f(-z)$ and n is even

Then $e^{-i\pi n} = 1 \Rightarrow a_n = 0$

(b) $\frac{|f(z)|}{|z|^n} \le M \quad (|z| \ge R)$; $C_R = \{z : |z| = R\}$

Then $|a_{n+m}| = \left| \frac{1}{2\pi i} \int_{C_R} \frac{f(\xi)}{\xi^{n+m+1}}\, d\xi \right|$

$$\le \frac{1}{2\pi} \cdot 2\pi R\, M \frac{1}{R^{m+1}} = \frac{M}{R^m} \xrightarrow[R\to\infty]{} 0 \quad (m \ge 1)$$

Therefore $a_{n+m} = 0$, $m \ge 1$

$\Rightarrow f(z)$ is a polynomial of degree $\le n$.