March 28, 2019 MATH 655 Prof. V. Panferov

Homework 8

Due on Wed. Apr 3, 2019.

1. If $f(z)$ is analytic for $0 < |z - z_0| < \rho$ and $|f(z)|$ is bounded, show that $f(z_0)$ can be defined in such a way that $f(z)$ becomes analytic for $|z - z_0| < \rho$.

   [Hint: Show that $h(z) = f(z)(z-z_0)^2$ is analytic for $|z-z_0| < \rho$ and look at its Taylor expansion.]

2. If $f(z)$ is analytic for $0 < |z - z_0| < \rho$ and $\mathrm{Re}(f(z))$ is bounded, show that $f(z_0)$ can be defined in such a way that $f(z)$ becomes analytic for $|z - z_0| < \rho$.

   [Hint: Without loss of generality, $0 \le \mathrm{Re}(f(z)) \le M$. Consider a fractional linear transformation $g(z)$ that maps the upper half-plane into the unit disk and use the result of the previous problem for the function $g(f(z))$.]

3. If u is harmonic and bounded for $0 < |z| < \rho$ show that the origin is a removable singularity in the sense that u becomes harmonic in $\{z : |z| < \rho\}$ when $u(0)$ is properly defined.

   [Hint: Use the result of Problem 6 in the previous homework set.]

4. (Problem 2-7:1) Obtain the power-series expansions for the following functions around the indicated points. Where the function is multiple-valued, give the results for all possible branches.

   (a) $(1 + z + z^2)^{-1}$, $z_0 = 0$ [Hint: use partial fractions; simplify the final result!]

   (b) $\sin^2 z$, $z_0 = 0$, $z_0 = -1$ [Hint: use the known series for $\cos(z)$, $z_0 = 0$.]

   (c) $z^{1/2}$, $z_0 = 1$, $z_0 = i\pi$

   (d) $\ln(iz + (1 - z^2)^{1/2})$, $z_0 = 0$, $z_0 = i$ [Hint: the series for the derivative is simpler.]

5. (Problem 2-7:2) In terms of the Bernoulli numbers B_n, show that

   (a) $\cot z = \dfrac{1}{z} \sum_{n=0}^{\infty} (-1)^n \dfrac{2^n B_{2n}}{(2n)!} z^{2n}$

   (b) $\tanh z = \sum_{n=1}^{\infty} \dfrac{2^{2n}(2^{2n} - 1)B_{2n}}{(2n)!} z^{2n-1}$

   (c) $(z - \pi)/(\sin z) = \sum_{n=0}^{\infty} (-1)^n \dfrac{2(2^{2n-1} - 1)B_{2n}}{(2n)!} (z - \pi)^{2n}$

[Hint: Reduce the functions above to a form that would allow to use the standard series

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

and notice that $B_{2k+1} = 0$ for $k = 1, 2, \ldots$.]

6. (Problem 2-7:3) Let $f(z)$ be analytic inside the unit circle, with $f(0) = 0$, and with $|f(z)| \leq 1$ inside and on the circle. Show that $|f(z)| \leq |z|$ for any point z inside the circle, and if equality holds for any interior point $z \neq 0$ then $f(z) = e^{i\alpha} z$ everywhere, with α some real constant (Schwarz's lemma).

7. (Problem 2-7:6) Show that:

   (a) The power-series expansion about the origin of an even (odd) analytic function contains only even (odd) powers of z.

   (b) If $f(z)$ is entire, with $f(z)/|z|^n$ bounded as $z \to \infty$, where n is some positive integer, then $f(z)$ must be a polynomial of degree $\leq n$.