HOMEWORK 7 — SOLUTIONS

#1. $u(x,y) = x^3 - 3xy^2$ — polynomial, so deriv. of any order are continuous.

$$u_x = 3x^2 - 3y^2 \qquad u_y = -6xy$$

$$u_{xx} = 6x \qquad u_{yy} = -6x$$

$$\Rightarrow \quad u_{xx} + u_{yy} = 0, \quad u \text{ is harmonic.}$$

method 1:

$$f'(z) = u_x - i u_y$$

$$= 3x^2 - 3y^2 + i\,6xy$$

$$= 3(x+iy)^2 = 3z^2$$

$$f(z) = z^3 = (x+iy)^3 =$$

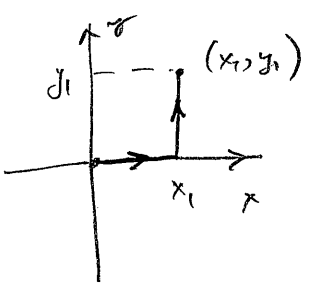
$$= x^3 - 3xy^2 + i(3yx^2 - y^3)$$

$$v(x,y) = 3yx^2 - y^3 \ (+C)$$

method 2.

$$u_x = v_y, \quad u_y = -v_x$$

$$v(x_1,y_1) = v_0 + \int_{(0,0)}^{(x_1,y_1)} u_x\,dy - u_y\,dx;$$



$$= v_0 + \int_{(0,0)}^{(x_1,y_1)} (3x^2 - 3y^2)\,dy + \underbrace{6xy\,dx}_{=0\ (y=0)}$$

$$= v_0 + 3x_1^2 y_1 - y_1^3$$

$$v(x,y) = 3x^2y - y^3 \ (+C).$$

<u>method 3</u>:

$$u_y = -6xy$$

(for each y:) $\quad v = -\int u_y \, dx = 3x^2 y + C(y)$

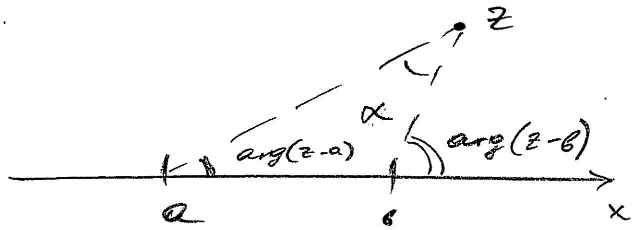
$$v_y = 3x^2 + C'(y) = u_x = 3x^2 - 3y^2$$

$$C'(y) = -3y^2$$

$$C(y) = -y^3 (+c)$$

$$v(x,y) = 3x^2 y - y^3 (+c).$$

#2.



$$\alpha = \arg(z-b) - \arg(z-a) = \arg\left(\frac{z-b}{z-a}\right) = \operatorname{Im} \log \frac{z-b}{z-a}$$

– harmonic as imaginary part of an analytic function.

Let $x \in \mathbb{R}$. If $x > b$ then

$$\left.\begin{array}{l} \arg(z-a) \to 0 \\ \arg(z-b) \to 0 \end{array}\right\} \text{ as } z \to x$$

If $x < a$ then

$$\left.\begin{array}{l} \arg(z-a) \to \pi \\ \arg(z-b) \to \pi \end{array}\right\} \text{ as } z \to x$$

$\Rightarrow$ $\alpha(z) \to 0$ as $z \to x$.

If $a < x < b$ then

$$\left.\begin{array}{l} \arg(z-a) \to 0 \\ \arg(z-b) \to \pi \end{array}\right\} \text{ as } z \to x \;\Rightarrow\; \alpha(z) \to \pi \text{ as } z \to x$$

Thus,

$$\lim_{z \to x} \alpha(z) = \begin{cases} 0, & x \notin (a,b) \\ \pi, & x \in (a,b) \end{cases}$$

The limit is undefined when $x = a, b$ since the value of $\arg(z-a), \arg(z-b)$ depend on the mode of approach.

#3. let u - continuous on U

satisfy $u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta$

for every disk $|\xi - z| \le r$.

Given $z_0 \in U$ and $r>0$ let w be the solution of the Dirichlet problem

$$\begin{cases} \Delta w = 0 & \text{in } |z-z_0|<r \\ w = u & \text{on } |z-z_0| = r \end{cases}$$

Then $g = u - w$ satisfies the mean-value property for every sub-disk of $|z-z_0| \le r$ and therefore must be constant on $|z-z_0| \le r$. This proves that u is harmonic on $|z-z_0|<r$. Since this holds for any disk in U, u must be harmonic on U.

#4.

(a) $$u(z_0) \le \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})\,d\theta$$

$$\Rightarrow u(z_0) \le \frac{1}{\pi r^2}\int_0^r \int_0^{2\pi} u(z+\rho e^{i\theta})\rho\,d\rho\,d\theta$$

$$= \frac{1}{\pi r^2}\iint_{|\xi-z|<r} u(\xi)\,dA_\xi .$$

If u achieves an interior max at $z_0 \in U$, then for some $r>0$ $\underset{|\xi-z_0|<r}{u(\xi)-u(z_0)\le 0}$.

$$u(z_0) = \frac{1}{\pi r^2}\iint_{|\xi-z_0|<r} u(z_0)\,dA_\xi \le \frac{1}{\pi r^2}\iint_{|\xi-z_0|<r} u(\xi)\,dA_\xi$$

$$\Rightarrow \iint_{|\xi-z_0|<r} u(\xi)-u(z_0)\,dA_\xi \ge 0$$

$$\Rightarrow \underbrace{u(\xi)-u(z_0)}_{\text{continuous!}} = 0 \quad \text{for } |\xi-z_0|\le r$$

Thus, $u(z)$ is constant on a neighborhood of a point of interior maximum

$\Rightarrow$ since any $z\in U$ can be connected to z_0 by a curve covered by disks $D_1 \dots D_n$ in which the center of D_{i+1} is contained in D_i then u must have the same value at z as at z_0 $\Rightarrow$ u is constant on U $\Rightarrow$ u achieves its maximum on the boundary.

(b)

$$u(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})\,d\theta$$

$$w(z_0) = \frac{1}{2\pi}\int_0^{2\pi} w(z+re^{i\theta})\,d\theta$$

$\Rightarrow$ $g = u - w$ is subharmonic

so $g \leq 0$ on Γ implies $g \leq 0$ in U.

(c)

$f(z)$ - analytic

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})\,d\theta$$

$$\Rightarrow |f(z)| = \frac{1}{2\pi}\left|\int_0^{2\pi} f(z+re^{i\theta})\,d\theta\right|$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} |f(z+re^{i\theta})|\,d\theta$$

$\Rightarrow$ $u(z) = |f(z)|$ is subharmonic.

#5. Divergence Theorem (Green's Theorem)

$$\iint_A \varphi_x + \psi_y \, dA = \int_\Gamma \varphi \, dy - \psi \, dx$$
$$= \int_\Gamma (\varphi, \psi) \cdot \left(\frac{dy}{ds}, -\frac{dx}{ds} \right) ds$$
$$= \int_\Gamma (\varphi, \psi) \cdot \vec{n} \, ds$$

outer normal

$$\iint_A \nabla u \cdot \nabla v + u \Delta v \, dA = \iint_A (u_x v_x + u_y v_y + u v_{xx} + u v_{yy}) \, dA$$

$$= \iint_A (u v_x)_x + (u v_y)_y \, dA = \int_\Gamma (u v_x, u v_y) \cdot \vec{n} \, ds$$

$$= \int_\Gamma u \, (v_x, v_y) \cdot \vec{n} \, ds = \int_\Gamma u \frac{\partial v}{\partial n} \, ds$$

Then

$$\int_\Gamma \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds = \iint_A \nabla u \cdot \nabla v \, dA + \iint_A u \Delta v \, dA$$
$$- \iint_A \nabla u \cdot \nabla v \, dA - \iint_A v \Delta u \, dA$$

$$= \iint_A (u \Delta v - v \Delta u) \, dA.$$

(a) If u is harmonic inside Γ then $\Delta u = 0$ in A, and using $v \equiv 1$,

$$\int_\Gamma \frac{\partial u}{\partial n} \, ds = \int_A \Delta u \, dA = 0.$$

(b) For any disk $|z-z_0| \le r$ in U and for any $0 < \rho \le r$ let $\Gamma_\rho = \{z : |z| = \rho\}$.

By using $u \equiv 1$ as in part (a), we get

$$\int_{\Gamma_\rho} \frac{\partial v}{\partial n} \, ds = \iint_{D(z_0,\rho)} \Delta v \, dA, \quad D(z_0,\rho) = \{z : |z-z_0| < \rho\}$$

Since $\frac{\partial v}{\partial n} = \frac{\partial v}{\partial \rho}$ (radial derivative)

and $ds = \rho \, d\vartheta$,

$$\int_{\Gamma_\rho} \frac{\partial v}{\partial n} \, ds = \int_0^{2\pi} \frac{\partial}{\partial \rho} v(z_0 + \rho e^{i\vartheta}) \, \rho \, ds$$

Multiply by $\frac{1}{\rho}$ and integrate over $(0, r)$:

$$\int_0^r \int_{\Gamma_\rho} \frac{\partial v}{\partial n} \, ds \, d\rho = \int_0^r \int_0^{2\pi} \frac{\partial}{\partial \rho} v(z_0 + \rho e^{i\vartheta}) \, \rho \, ds \, \frac{1}{\rho} \, d\rho$$

$$= \int_0^{2\pi} \int_0^r \frac{\partial}{\partial \rho} v(z_0 + \rho e^{i\vartheta}) \, d\rho \, ds = \int_0^{2\pi} \left(v(z_0 + re^{i\vartheta}) - v(z_0) \right) ds$$

$$= \int_0^{2\pi} v(z_0 + re^{i\vartheta}) \, d\vartheta - 2\pi v(z_0) \ge 0$$ since v is subharmonic.

Therefore

$$\int_0^r \iint_{D(z_0,\rho)} \Delta v \, dA \, \frac{1}{\rho} \, d\rho = \int_0^r \int_0^{2\pi} \int_0^\rho \Delta v(z_0 + s e^{i\vartheta}) \, s \, ds \, d\vartheta \, \frac{1}{\rho} \, d\rho$$

$$= \int_0^{2\pi} \int_0^r \Delta v(z_0 + s e^{i\vartheta}) \, s \underbrace{\left(\int_s^r \frac{1}{\rho} \, d\rho \right)}_{= \ln \frac{r}{s}} ds \, d\vartheta$$

$$= \iint_{D(z_0, r)} \Delta v \, \ln \frac{r}{|\xi - z_0|} \, dA_\xi \ge 0$$

Since z_0 and $r > 0$ are arbitrary, it follows that $\Delta v \ge 0$. Indeed, if $\Delta v < 0$ for a $z_0 \in U$, choose r small enough so $\Delta v < 0$ for $|z - z_0| < r$

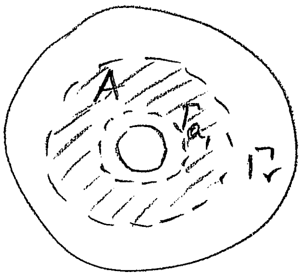
then $$\iint_{D(z_0, r)} \underbrace{\Delta v}_{<0} \, \underbrace{\ln \frac{r}{|\xi - z_0|}}_{\ge 0} \, dA_\xi < 0$$ in contradiction to the above inequality.

#6. Let $U = \{z : r_1 < |z| < r_2\}$ and

$A = \{z : a_1 < |z| < r\}$ with

$r_1 < a_1 < r < r_2.$

The integral identities of the previous problem extend to the case of annulus A:



$$\int_{\Gamma} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds = \iint_A \underbrace{(u \Delta v - v \Delta u)}_{=0} \, dA$$

if u, v are harmonic:

and $\Gamma = \Gamma_{a_1} \cup \Gamma_r$

$\Gamma_{a_1} = \{z : |z| = a_1\}$; $\Gamma_r = \{z : |z| = r\}$.

on Γ_{a_1}, $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$ while on Γ_r, $\frac{\partial}{\partial n} = \frac{\partial}{\partial r}$

Using $v = 1$ we get

$$-\int_{\Gamma_{a_1}} \frac{\partial u}{\partial r} \, ds + \int_{\Gamma_r} \frac{\partial u}{\partial r} \, ds = 0$$

Thus, for $r > a_1$

$$\int_{\Gamma_r} \frac{\partial u}{\partial r} \, ds = \alpha_0 = \text{constant}.$$

Since a_1 is arbitrary, the same holds for all $r \in (r_1, r_2)$.

Using $v = \log|z|$,

$$-\int_{\Gamma_{a_1}} \left(u \frac{1}{a_1} - \log a_1 \frac{\partial u}{\partial r} \right) ds + \int_{\Gamma_r} \left(u \frac{1}{r} - \log r \frac{\partial u}{\partial r} \right) ds = 0$$

so

$$\int_{\Gamma_r} u \frac{1}{r} \, ds - \log r \int_{\Gamma_r} \frac{\partial u}{\partial r} \, ds = \beta_0 = \text{constant}.$$

Since $\int_{\Gamma_r} u \frac{1}{r}\, ds = \int_0^{2\pi} u(re^{i\theta})\, d\theta$

$$\Rightarrow \quad \int_0^{2\pi} u(re^{i\theta})\, d\theta = \alpha_0 \log r + \beta_0$$

$$\Rightarrow \quad \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta})\, d\theta = \frac{\alpha_0}{2\pi} \log r + \frac{\beta_0}{2\pi}$$

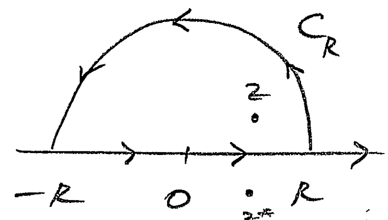
$$= \alpha \log r + \beta.$$

#7. Dirichlet problem for the upper half-plane:

$$\begin{cases} \Delta u = 0 & \text{in } U = \{z : \operatorname{Im}(z) > 0\} \\ u = \varphi & \text{on } \Gamma = \mathbb{R} \end{cases}$$

Suppose also that $f(z) = u + iv$ is bounded on U.

Cauchy's Integral Formula:



$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\xi)}{\xi - z}\, d\xi$$

$$0 = \frac{1}{2\pi i} \int_{C_R} \frac{f(\xi)}{\xi - z^*}\, d\xi \qquad \left[\begin{array}{l} z \in \operatorname{Int}(C_R) \\ \Rightarrow z^* \notin \operatorname{Int}(C_R) \end{array}\right]$$

$$f(z) = \frac{1}{2\pi i} \int_{C_R} f(\xi) \left(\frac{1}{\xi - z} \pm \frac{1}{\xi - z^*} \right) d\xi$$

The "$-$" sign gives $\qquad (z = x + iy)$

$$\frac{1}{\xi - z} - \frac{1}{\xi - z^*} = \frac{\xi - z^* - \xi + z}{(\xi - z)(\xi - z^*)} = \frac{2yi}{(\xi - z)(\xi - z^*)}$$

Fix $z \in U$ and let $R \to \infty$. Let $C_R = [-R, R] \cup \underbrace{C_R^+}_{\text{half circle}}$

$$\left| \frac{1}{2\pi i} \int_{C_R^+} f(\xi) \frac{2y}{(\xi - z)(\xi - z^*)}\, d\xi \right|$$

$$\leq \frac{|y|}{\pi} \int_{C_R^+} \underbrace{|f(\xi)|}_{\leq M} \frac{1}{(R - |z|)^2} |d\xi|$$

$$\leq \frac{|y| M}{\pi} \cdot \underbrace{\pi R}_{\text{length}(C_R^+)} \cdot \frac{1}{(R - |z|)^2} \xrightarrow[R \to \infty]{} 0.$$

Thus

$$f(z) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{C_R} f(\xi) \left(\frac{1}{\xi - z} - \frac{1}{\xi - z^*} \right) d\xi$$

$$= \lim_{R\to\infty} \frac{1}{2\pi i} \int_{-R}^{R} f(\xi) \frac{2yi}{\underbrace{|\xi - z|^2}} d\xi$$

since $\xi \in \mathbb{R} \Rightarrow (\xi - z)(\xi - z^*) = |\xi - z|^2$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \frac{y}{(\xi - x)^2 + y^2} d\xi$$

(The limit exists since the contour integral has the same value for all R large enough and the integral over $C_R^+ \to 0$.)