March 15, 2019 MATH 655 Prof. V. Panferov

Homework 7

Due on Wed. Mar 27, 2019.

1. Let $u(x, y) = x^3 - 3xy^2$. Show that u is harmonic in $\mathbb{R}^2$ and find its harmonic conjugate.

2. Prove that the visual angle $\omega(z)$ of the segment $[a, b]$ on the real axis, measured from a point z of the upper half-plane of $\mathbb{C}$, is a harmonic function in the upper half-plane. Find $\lim\limits_{z \to x} \omega(z)$ for x on the real axis.

3. (Problem 2-5:4) Show that if a continuous real function $u(x, y)$ satisfies the mean-value theorem for every circle in the region, then that function is harmonic in the region. *[Hint: Consider a disk $|z - z_0| \le r$ within the region and define w to be the solution of the Dirichlet problem in this disk with the boundary data given by u. Then apply the maximum principle to the difference $g = u - w$.]*

4. (Problem 2-5:6) Let U be a bounded region in $\mathbb{R}^2$ and denote by $\overline{U}$ its closure in $\mathbb{R}^2$ and by Γ its boundary. Suppose $u : \overline{U} \to \mathbb{R}$ is continuous and satisfies the inequality

$$u(z_0) \le \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta$$

   for any $z_0 \in U$ and for circles of any radius r lying wholly within U. Such a function is said to be *subharmonic*. Show the following:

   (a) The maximum of u over $\overline{U}$ must occur on the boundary of U.

   (b) If $w : \overline{U} \to \mathbb{R}$ is continuous and harmonic in U and u is subharmonic such that $u \le w$ on Γ then $u \le w$ in U.

   (c) The modulus of any analytic function is subharmonic.

5. (Problem 2-5:7) Let the functions u and v have continuous derivatives of order two in a region $U \subseteq \mathbb{R}^2$, and let Γ denote a closed contour in U and A the region inside it. Using Green's theorem obtain the identities

$$\int_\Gamma u \frac{\partial v}{\partial n}\, ds = \iint_A \nabla u \cdot \nabla v\, dA + \iint_A u \Delta v\, dA$$

   and

$$\int_\Gamma \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds = \iint_A (u\Delta v - v \Delta u)\, dA,$$

   where $\nabla = (\partial/\partial x, \partial/\partial y)$, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and $\partial/\partial n$ is the directional derivative along the outer normal on the boundary. Use these identities to show the following:

(a) For any function u harmonic inside a contour Γ

$$\int_\Gamma \frac{\partial u}{\partial n}\, ds = 0.$$

(b) A smooth function v (twice differentiable, with continuous derivatives) is subharmonic if and only if $\Delta v \geq 0$.

[Hint for (b): Set $u = 1$ and let Γ be a circle.]

6. Suppose u is harmonic in an annulus $r_1 < |z| < r_2$. Show that for $r \in (r_1, r_2)$

$$\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\, d\theta = \alpha \ln r + \beta,$$

for some real constants α and β.

[Hint: Let $U = \{z : r_1 < |z| < r_2\}$ and $A = \{z : a_1 < |z| < r\}$ such that $r_1 < a_1 < r < r_2$. Notice that the integral identities in the previous problem still apply to the multiply connected region A, with the boundary Γ consisting of the inner and outer circles $\{|z| = a_1\}$ and $\{|z| = r\}$. On the inner circle $\partial/\partial n = -\partial/\partial r$, while on the outer circle $\partial/\partial n = \partial/\partial r$. Use the second integral identity with $v = 1$ to show that

$$\int_{|z|=r} \frac{\partial u}{\partial r}\, ds = \alpha_0 = \text{constant}.$$

Then use $v = \ln|z|$ in the same identity to obtain the desired result.]

7. Apply the method used to derive Poisson's integral formula for the unit circle (p. 47) to obtain the following Poisson's formula for the solution of the Dirichlet problem in the upper half-plane:

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y\varphi(\xi)}{(x-\xi)^2 + y^2}\, d\xi.$$

Assume that the corresponding analytic function $f(z) = u + iv$ is bounded in the upper half-plane.

[Hint: Use Cauchy's Integral Formula for a contour C_R composed of the interval $[-R, R]$ and the upper half-circle $|z| = R$, $\mathrm{Im}(z) > 0$. Use the facts that for z in the upper half-plane

$$\int_{C_R} \frac{f(\zeta)}{\zeta - z^*}\, d\zeta = 0$$

and

$$\frac{1}{2\pi i}\left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z^*}\right) = \frac{1}{\pi}\frac{y}{(\zeta - z)(\zeta - z^*)}.$$

Take the limit $R \to \infty$.]