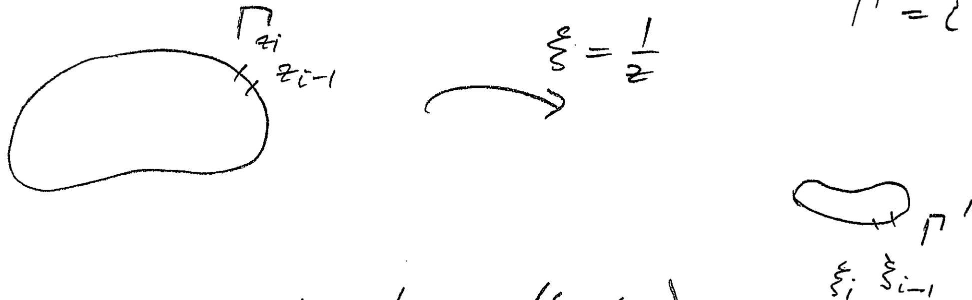
#1. Yes, the change of variables $z = 1/\xi$ is OK when Γ is a closed contour, $\Gamma \not\ni 0$.

$$\Gamma' = \{\xi : 1/\xi \in \Gamma\}$$



$$\Delta z_i = z_i - z_{i-1} = \frac{1}{\xi_i} - \frac{1}{\xi_{i-1}} = \frac{-(\xi_i - \xi_{i-1})}{\xi_i\,\xi_{i-1}} \qquad (|\Delta z| = \max|\Delta z_i|)$$

So
$$\int_\Gamma f(z)\,dz = \lim_{|\Delta z|\to 0} \sum_{i=1}^{n} f(z_i)\,\Delta z_i$$
$$= \lim_{|\Delta\xi|\to 0} \sum_{i=1}^{n} f\left(\frac{1}{\xi_i}\right)\left(-\frac{1}{\xi_i\xi_{i-1}}\right)\Delta\xi_i$$
$$= \lim_{|\Delta\xi|\to 0} \sum_{i=1}^{n} f\left(\frac{1}{\xi_i}\right)\left(-\frac{1}{\xi_i^2}\right)\Delta\xi_i$$
$$= \int_{\Gamma'} f\left(\frac{1}{\xi}\right)\left(-\frac{1}{\xi^2}\right)d\xi$$

where the contour Γ' is traveled in the clock wise (opposite) direction.

So
$$\int_\Gamma \frac{dz}{1+z^3} = \int_{\Gamma'} \frac{-\frac{1}{\xi^2}\,d\xi}{1+1/\xi^3} = \int_{\Gamma'} \frac{-\xi\,d\xi}{\xi^3+1} = 0$$

since the roots of ξ^3+1 are outside
$$\Gamma' = \{\xi : |\xi| = \tfrac{1}{2}\}.$$

$(n > 2 \Rightarrow n-2 \geq 0!)$ ✓ 2

Similarly,

$$\int_\Gamma \frac{dz}{P_n(z)} = \int_{\Gamma'} \frac{-\frac{1}{\xi^2}d\xi}{P_n\left(\frac{1}{\xi}\right)} = \int_{\Gamma'} \frac{-\xi^{n-2}d\xi}{Q_n(\xi)}$$

where Q_n is a polynomial of degree $\leq n$ with all roots outside the contour Γ'.

Therefore $\displaystyle\int_\Gamma \frac{dz}{P_n(z)} = 0$ for any contour Γ containing all the zeros of P_n.

#2: Assume $f(z) \to f(\infty) = \alpha$ as $z \to \infty$.

Then $f\left(\frac{1}{w}\right) \to f(\infty)$ as $w \to 0$.

Assume 0 is inside Γ.

(If not use $w = \frac{1}{z-a}$, for a inside Γ, with same result)

$$2\pi i\, f(z) = \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad ; \quad \zeta = 1/\xi, \quad z = 1/w$$

$$2\pi i\, f\left(\frac{1}{w}\right) = \int_{\Gamma'} \frac{-f\left(\frac{1}{\xi}\right)\frac{1}{\xi^2}\, d\xi}{1/\xi - 1/w}$$

$$2\pi i\, f\left(\frac{1}{w}\right) = \int_{-\Gamma'} \frac{w\xi\, f\left(\frac{1}{\xi}\right)\frac{1}{\xi^2}\, d\xi}{w - \xi} = \int_C f\left(\frac{1}{\xi}\right) \frac{w\, d\xi}{\xi(w-\xi)}$$

$$\frac{w}{\xi(w-\xi)} = \frac{1}{\xi} + \frac{1}{w-\xi} = \frac{1}{\xi} - \frac{1}{\xi - w} \qquad \left[\begin{array}{c} C = -\Gamma' \\ \text{counter-clockwise} \end{array}\right]$$

$$2\pi i\, f\left(\frac{1}{w}\right) = \int_C f\left(\frac{1}{\xi}\right)\frac{1}{\xi}\, d\xi - \int_C f\left(\frac{1}{\xi}\right)\frac{1}{\xi - w}\, d\xi$$

Now $\int_C f\left(\frac{1}{\zeta}\right)\frac{1}{\zeta}\,d\zeta = \lim_{\varepsilon\to 0}\int_0^{2\pi} f\left(\frac{1}{\varepsilon}e^{-i\theta}\right) i\,d\theta$

$$= 2\pi i\, f(\infty)$$

and so

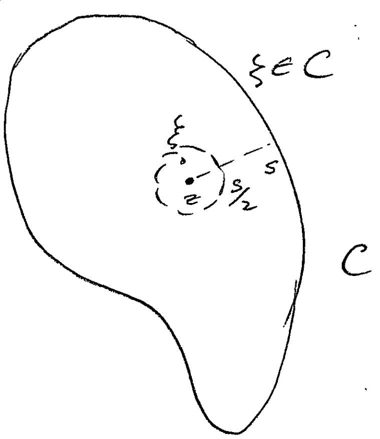
$$2\pi i\, f\left(\frac{1}{w}\right) = 2\pi i\, f(\infty) - \int_C f\left(\frac{1}{\zeta}\right)\frac{1}{\zeta - w}\,d\zeta$$

Thus, for $f(z)$ analytic outside contour C,

$$f(z) = f(\infty) - \frac{1}{2\pi i}\int_C f(\zeta)\,\frac{1}{\zeta - z}\,d\zeta$$

#3. Let $z \in \mathbb{C}$ be a point inside C.

$$\frac{f(\xi)-f(z)}{\xi - z} = \frac{1}{2\pi i}\int_C \varphi(\zeta)\left(\frac{1}{\zeta-\xi} - \frac{1}{\zeta - z}\right)\frac{1}{\xi - z}\,d\zeta$$



$$= \frac{1}{2\pi i}\int_C \varphi(\zeta)\,\frac{\cancel{(\xi - z)}}{(\zeta-\xi)(\zeta - z)}\,\frac{1}{\cancel{(\xi - z)}}\,d\zeta$$

$$= \frac{1}{2\pi i}\int_C \varphi(\zeta)\,\frac{1}{(\zeta - z)^2}\left(1 + \frac{\xi - z}{\zeta - \xi}\right)d\zeta$$

Let $s = \operatorname{dist}(z, C)$, $|\xi - z| < \delta \le \frac{s}{2}$

Then $\left|\frac{\xi - z}{\zeta - \xi}\right| < \frac{\delta}{s/2} = \frac{2}{s}\delta$

So the limit of the last integral as $\xi \to z$

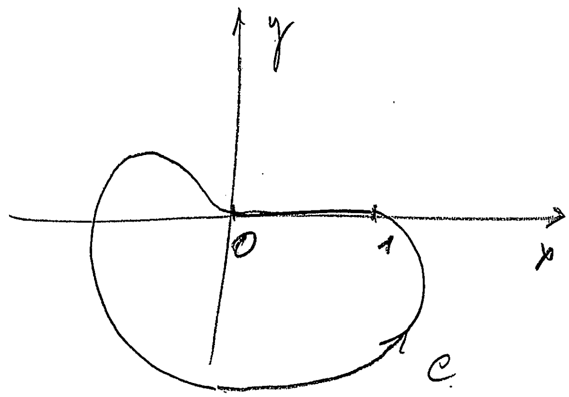
is $\frac{1}{2\pi i}\int_C \varphi(\zeta)\frac{1}{(\zeta - z)^2}\,d\zeta$ — well-defined since the integrand is continuous on C.

$\Rightarrow$ $f(z)$ is analytic.

To obtain an example when $f(z) \underset{z \to z_0}{\not\to} \varphi(z_0)$

take
$$\varphi(z) = \begin{cases} z(1-z), & z \in [0, 1] \\ 0 ; & \text{everywhere else.} \end{cases}$$

and consider C as follows:



Then for z inside C,

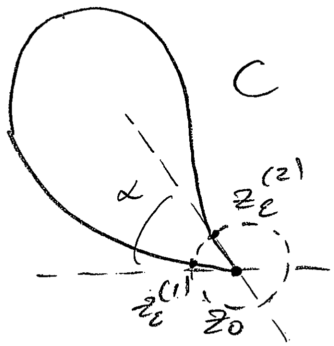
$$f(z) = \frac{-1}{2\pi i} \int_0^1 \frac{\varphi(x)\,dx}{x - z}.$$

Take $z = -\varepsilon$; $\varepsilon > 0$

Then
$$\lim_{\varepsilon \to 0} f(-\varepsilon) = \lim_{\varepsilon \to 0} -\frac{1}{2\pi i} \int_0^1 \frac{\varphi(x)\,dx}{x + \varepsilon}$$
$$= -\frac{1}{2\pi i} \int_0^1 \frac{\varphi(x)\,dx}{x}$$
$$= -\frac{1}{2\pi i} \int_0^1 (1-x)\,dx = \frac{i}{4\pi} \neq 0$$

while $\varphi(0) = 0$.

#4.



$$\text{p.v. } \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-z_0}$$

$$= \lim_{\varepsilon\to 0} \frac{1}{2\pi i}\int_{C_\varepsilon} \frac{f(z)\,dz}{z-z_0}$$

where $C_\varepsilon = C \setminus \underbrace{B(z_0,\varepsilon)}_{\text{disk of radius } \varepsilon \text{ centered at } z_0.}$

$$= \lim_{\varepsilon\to 0} \frac{1}{2\pi}\int_{\alpha(\varepsilon)}^{\beta(\varepsilon)} f(z_0+\varepsilon e^{i\theta})\,d\theta$$

where $\alpha(\varepsilon)$ and $\beta(\varepsilon)$ are two angles such that

$$\left.\begin{array}{l} z_\varepsilon^{(1)} = z_0+\varepsilon e^{i\alpha(\varepsilon)} \\ \text{and } z_\varepsilon^{(2)} = z_0+\varepsilon e^{i\beta(\varepsilon)} \end{array}\right\} \in C$$

(see figure.)

As $\varepsilon\to 0$ we have $\beta(\varepsilon)-\alpha(\varepsilon)\to\alpha$

and $f(z_0+\varepsilon e^{i\theta}) \longrightarrow f(z_0)$.

Thus $\displaystyle\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{\alpha(\varepsilon)}^{\beta(\varepsilon)} f(z_0+\varepsilon e^{i\theta})\,d\theta$

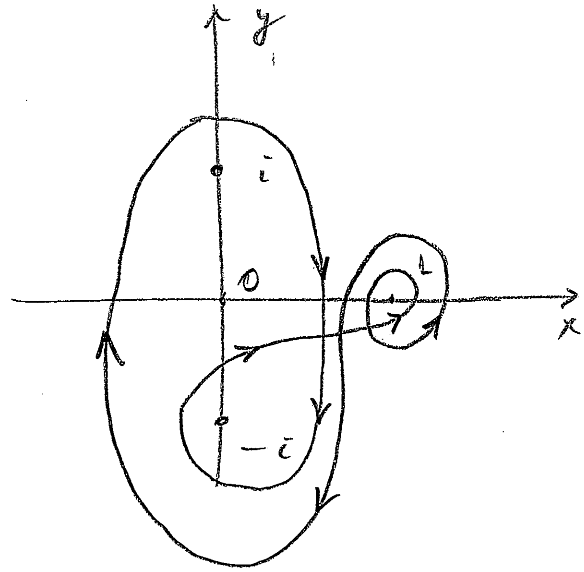
$$= \frac{\alpha}{2\pi} f(z_0).$$

#5. (a)

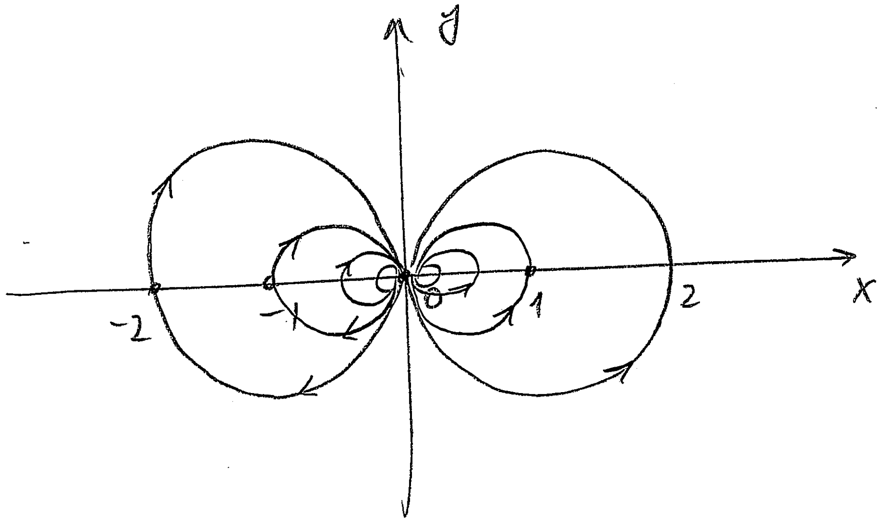
$w(\gamma, i) = -1$

$w(\gamma, 1) = 2$

$w(\gamma, -i) = -2$



(b)



The path consists of circles of radius $2^{-(k-1)}$, $k = 1, 2, \ldots$ centered about $z_k = \pm 2^{-(k-1)}$

The circles on the positive half-plane are oriented counterclockwise while the circles on the negative half-plane are oriented clockwise.

The total length of the path is

$$2 \cdot \sum_{k=1}^{\infty} 2\pi \cdot 2^{-(k-1)} = 8\pi \sum_{k=1}^{\infty} 2^{-k} = 8\pi$$

(finite!)

Parametrization:

Set $$z(t) = 2^{-(k-1)}\left(1 - e^{i 2^{(k-1)} t}\right),$$

$$t \in [s_{k-1}, s_k)$$

where $s_0 = 0$, $s_k = \sum_{j=1}^{k} 2\pi \cdot 2^{-(k-1)} = 4\pi(1-2^{-k})$

Also
$$z(t) = 2^{-(k-1)}\left(-1 + e^{-i 2^{(k-1)} t}\right)$$

$$t \in [-s_k, -s_{k-1})$$

Notice that when $t = \pm s_k$ $\quad z(t) = 0$

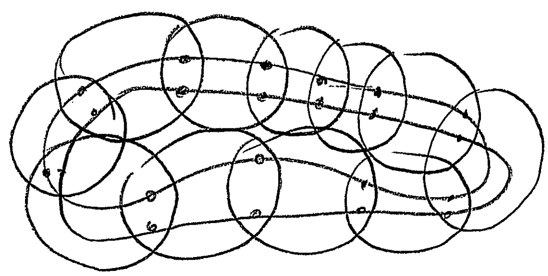
$\quad z'(t) = -i$.

$\Rightarrow z(t)$ is differentiable on $(-4\pi, 4\pi)$.

with $z(-4\pi) = z(4\pi) = 0$ $\quad z(t)$ is also continuous on $[-4\pi, 4\pi]$.

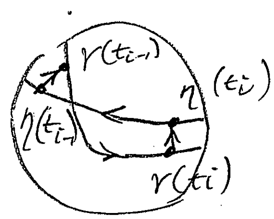
#6. Suppose $\gamma, \eta : [a, b] \to \mathbb{C}$, closed paths are close together in U.

Pick $a = t_0 < t_1 < \ldots < t_n = b$ and cover γ, η by disks D_i, $i = 1 \ldots n$.



Connect $\gamma(t_i), \eta(t_i)$ by straight line segments

Let $f(z) = \frac{1}{z - \alpha}$, $\alpha \notin U$



Since f is holomorphic in D_i

$$\int_{C_i} f(z)\,dz = 0, \text{ where}$$

$$C_i : \gamma(t_{i-1}) \to \gamma(t_i) \to \eta(t_i) \to \eta(t_{i-1}) \to \gamma(t_{i-1})$$

Since this is true for each $i = 1 \ldots n$, then

$$0 = \sum_{i=1}^{n} \int_{C_i} f(z)\,dz = \sum_{i=1}^{n} \left(\int_{\gamma(t_{i-1})}^{\gamma(t_i)} f(z)\,dz + \int_{\gamma(t_i)}^{\eta(t_i)} f(z)\,dz + \int_{\eta(t_i)}^{\eta(t_{i-1})} f(z)\,dz + \int_{\eta(t_{i-1})}^{\gamma(t_{i-1})} f(z)\,dz \right)$$

$$= \int_\gamma f(z)\,dz - \int_\eta f(z)\,dz = W(\gamma, \alpha) - W(\eta, \alpha).$$

telescope

Since $\alpha \notin U$ is arbitrary, γ and η are homologous.

#7. Suppose $\gamma, \eta : [a,b] \to \mathbb{C}$, closed paths are homotopic in U.

$\psi : \underbrace{[a,b) \times [c,d]}_{=S} \to \mathbb{C}$: continuous,

$\psi(\cdot, c) = \gamma$; $\psi(\cdot, d) = \eta$;

$\forall s \in [c,d]$ $\psi(\cdot, s)$ is a closed path.

$\psi(S)$ is compact in U

$\Rightarrow \exists \varepsilon_1 > 0 : \quad B(z, \varepsilon_1) \subseteq U \quad \forall z \in \psi(S)$.

Choose $\delta_1 > 0$ so that $|t_1 - t_2| < \delta_1$, $|s_1 - s_2| < \delta_1$

$\Rightarrow \psi(t_1, s_1), \psi(t_1, s_2), \psi(t_2, s_1), \psi(t_2, s_2) \in B(z, \varepsilon_1)$

for some $z \in \psi(S)$.

Pick subdivisions

$a = t_0 < t_1 < \dots < t_n = b$

$c = s_0 < s_1 < \dots < s_m = d$

with $\max (t_i - t_{i-1}) < \delta_1$

$\max (s_j - s_{j-1}) < \delta_1$.

Then each pair of paths

$\psi(\cdot, s_j), \psi(\cdot, s_{j-1})$

is close together in U

since $\psi([t_{i-1}, t_i], s_j), \psi([t_{i-1}, t_i], s_{j-1})$

$\subseteq B(z_{ij}, \varepsilon_1)$ by the choice of δ_1.

for some $z_{ij} \in \psi(S)$.

Let $f(z) = \dfrac{1}{z - \alpha}$, $\alpha \notin U$.

Since f is holomorphic in $B(z_{ij}, \varepsilon_1)$

$$\int_{\psi(\cdot, s_j)} f(z)\,dz = \int_{\psi(\cdot, s_{j-1})} f(z)\,dz.$$

Therefore $\forall \alpha \in U$,

$$W(\gamma, \alpha) = \int_{\psi(\cdot, s_0)} f(z)\,dz = \int_{\psi(\cdot, s_m)} f(z)\,dz$$

$$= W(\eta, \alpha)$$

$\Rightarrow$ γ and η are homologous on U.

#8. Let U be the region bounded by the contour C; $\overline{U} = U \cup C$.

$\overline{U}$ is closed and bounded, so the continuous function $|f(z)|$ must achieve a minimum on $\overline{U}$.

If this minimum is positive, then $\left|\frac{1}{f(z)}\right|$ achieves a finite maximum on $\overline{U}$ which is impossible since $\left|\frac{1}{f(z)}\right|$ does not achieve max on the boundary, and $\left|\frac{1}{f(z)}\right|$ cannot achieve a maximum in U since $\frac{1}{f(z)}$ is analytic on U. Therefore, the minimum of $|f(z)|$ on U must be zero.

Now let $f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$, $a_n \neq 0$ be a non-constant polynomial.

Take $C = C_R$ - the circle $|z| = R$, with R sufficiently large. Then $|f(0)| = |a_0|$, and for $z \in C_R$

$$|f(z)| = \left|a_n z^n \left(1 + \frac{a_{n-1}}{a_n}\frac{1}{z} + \dots + \frac{a_0}{a_n}\frac{1}{z^n}\right)\right|$$

$$\geq |a_n| R^n \left(1 - \left|\frac{a_{n-1}}{a_n}\right|\frac{1}{R} - \dots - \left|\frac{a_0}{a_n}\right|\frac{1}{R^n}\right)$$

$$> |a_0| + 1 \text{ for } R \text{ sufficiently large.}$$

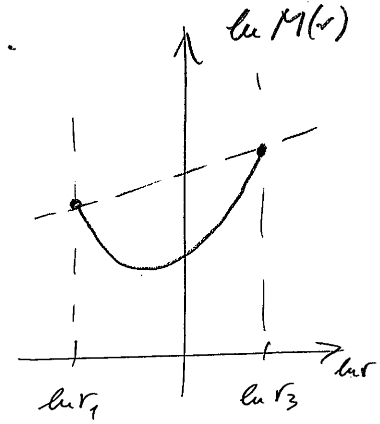
Thus, we can apply the first part of the problem with $M = |a_0| + 1$ and $z_0 = 0$.

\#9. Consider $w(z) = z^s f(z)$, $s \in \mathbb{R}$.

$w(z)$ is analytic on $\{z : r_1 < |z| < r_3\} \setminus [-r_3, -r_1]$, therefore $|w(z)|$ achieves its maximum on the boundary. However, the maximum cannot be achieved on $[-r_3, -r_1]$, since $|w(z)|$ is continuous on $[-r_3, -r_1]$, and if it had a maximum there, then another branch of $w(z)$ had an interior maximum. Thus, the maximum of $|w(z)|$ must be achieved for $|z| = r_1$ or $|z| = r_3$.

We have

$$\ln |w(z)| = s \ln |z| + \ln |f(z)|.$$



so

$$\max_{|z|=r} \ln |w(z)| = s \ln r + \ln M(r).$$

Choose

$$s = -\frac{\ln M(r_3) - \ln M(r_1)}{\ln r_3 - \ln r_1}$$

Then

$$\max_{|z|=r_3} \ln |w(z)| - \max_{|z|=r_1} \ln |w(z)| = \ln M(r_3) - \ln M(r_1) - \frac{\ln M(r_3) - \ln M(r_1)}{r_3 - r_1}(r_3 - r_1) = 0$$

So

$$\ln|w(z)| \le \ln M(r_1) - \frac{\ln M(r_3) - \ln M(r_1)}{\ln r_3 - \ln r_1} \ln r_1$$

$$\Rightarrow \quad \ln M(r_2) \le \ln M(r_1) + \frac{\ln M(r_3) - \ln M(r_1)}{\ln r_3 - \ln r_1} (\ln r_2 - \ln r_1)$$

$$\Rightarrow \quad \ln M(r_2) \le \ln M(r_1) + t\left(\ln M(r_3) - \ln M(r_1)\right),$$

$$t = \frac{\ln r_2 - \ln r_1}{\ln r_3 - \ln r_1}$$

$$\Rightarrow \quad \ln M(r_2) \le (1-t) \ln M(r_1) + t \ln M(r_3)$$

$$\Rightarrow \quad \ln M(r_2) \le \frac{\ln r_3 - \ln r_2}{\ln r_3 - \ln r_1} \ln M(r_1) + \frac{\ln r_2 - \ln r_1}{\ln r_3 - \ln r_1} \ln M(r_3).$$

<u>Note</u>: The key is the following one-dimensional lemma:

If $\varphi(x)$ is such that $\forall s \in \mathbb{R}$

$$\max_{[a,b]} \varphi(x) + sx = \max\{\varphi(a)+sa, \varphi(b)+sb\}$$

then $\varphi(x)$ is convex on $[a,b]$.

For a proof, choose $s = -\dfrac{\varphi(b)-\varphi(a)}{b-a}$

then

$$\varphi(x) \le \varphi(a) + \frac{\varphi(b)-\varphi(a)}{b-a}(x-a).$$

$$\Rightarrow \varphi(a(1-t)+bt) \le \varphi(a)(1-t) + \varphi(b)t.$$