Homework 6

Due on Wed. Mar 13, 2019.

1. (Problem 2-2:7) Carry out the change of variable $z = 1/\xi$ in order to evaluate

$$\int_\Gamma \frac{dz}{1+z^3}$$

where Γ is a counterclockwise path around the circle $|z| = 2$. Is it legitimate to write $dz = -(1/\xi^2)\,d\xi$? What is the new contour, what are the properties of integrand within it, and is the new sense of integration clockwise or counterclockwise? Obtain a similar result for the integral of $1/P_n(z)$ around any contour containing all the zeros of the n-th order polynomial $P_n(z)$ if $n \geq 2$.

2. (Problem 2-3.5) Obtain a version of Cauchy's Integral Formula valid for the situation in which $f(z)$ is analytic everywhere *outside* some closed curve C, with $f(z) \to$ some constant α as $z \to \infty$. Relate this result to the transformation $\zeta = 1/z$.

3. (Problem 2-3.6) Show, by the method used to derive equation 2-13, that if $\Phi(z)$ is any function continuous on C (but not necessarily analytic there or anywhere else), and if a function $f(z)$ is defined for z inside C by

$$f(z) = \frac{1}{2\pi i}\int_C \frac{\Phi(\zeta)}{\zeta - z}\,d\zeta$$

then $f(z)$ is analytic inside C. Give an example to illustrate the fact that, as an interior point z approaches a boundary point z_0, $f(z)$ need not approach $\Phi(z_0)$.

4. Verify, as stated in the textbook on p. 39, that if piecewise smooth closed contour C has a corner point at z_0 with an interior angle $\alpha \in (0, 2\pi)$ and $f(z)$ is holomorphic in a region containing C with its interior then

$$\text{p.v.}\ \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz = \frac{\alpha}{2\pi} f(z_0).$$

5. (a) Draw an example of a smooth closed path γ in $\mathbb{C}$ such that $W(\gamma, 1) = 2$, $W(\gamma, i) = -1$ and $W(\gamma, -i) = -2$.

   (b) Give an example of a piecewise smooth closed path γ of finite length in $\mathbb{C}$ such that for any integer k there is a point $\alpha \notin \gamma$ with $W(\gamma, \alpha) = k$. Specify a parametrization of γ explicitly.

6. (Lang, Theorem 4.2.1 (ii)) Let U be an open subset of $\mathbb{C}$. We say that two paths γ and $\eta : [a, b] \to \mathbb{C}$ are *close together* in U if "there exists a common covering of γ and η by discs in U". More precisely, for a certain partition $a = t_0 < \cdots < t_n = b$ of the interval $[a, b]$, and for each $i \in \{1, \ldots, n\}$ there exists a disc $D_i \subseteq U$ such that $\gamma([t_{i-1}, t_i]) \subseteq D_i$ and $\eta([t_{i-1}, t_i]) \subseteq D_i$. Prove that if two closed paths are close together in U then they are homologous in U.

7. (Lang, Theorem 4.2.1 (i)) Let U be an open subset of $\mathbb{C}$. We say that two closed paths γ and $\eta : [a, b] \to \mathbb{C}$ are *homotopic* in U if there exists a continuous function $\psi : [a, b] \times [c, d] \to U$ such that $\psi(\cdot, c) = \gamma$, $\psi(\cdot, d) = \eta$ and for each $s \in [c, d]$ $\psi(\cdot, s)$ is a closed path. Prove that if two closed paths are homotopic in U then they are homologous in U.

8. (Problem 2-4.3) Let the real number $M > 0$ be a lower bound for $|f(z)|$ on some contour C within which $f(z)$ is holomorphic. Let z_0 be a point within C, and suppose that $|f(z_0)| < M$. Show that $f(z)$ must have at least one zero inside C. Use this result to prove that every nonconstant polynomial must have a zero in $\mathbb{C}$.

9*. (Problem 2-4.6) Let $f(z)$ be analytic and single-valued in the ring-shaped region $R = \{z : r_1 \leq |z| \leq r_3\}$. Let r_2 be any number satisfying $r_1 < r_2 < r_3$. Denote the maximum of $|f(z)|$ on the circle $|z| = r_j$ by $M(r_j)$, $j = 1, 2, 3$. Then prove Hadamard's Three Circles Theorem, which states that $\ln M(r)$ is a convex function of $\ln r$, in the sense that

$$\ln M(r_2) \leq \frac{\ln r_3 - \ln r_2}{\ln r_3 - \ln r_1} \ln M(r_1) + \frac{\ln r_2 - \ln r_1}{\ln r_3 - \ln r_1} \ln M(r_3).$$

[See hint in the textbook.]