**#1**: Mean-value Theorem: $\dfrac{f(b)-f(a)}{b-a} = f'(c)$.

works when $f$ is constant; linear:
(trivial) $\dfrac{(mb+k)-(ma+k)}{b-a} = m = f'(c)$

quadratic: $\dfrac{b^2-a^2}{b-a} = b+a = f'(c)$ where $c = \frac{b+a}{2}$.

Try cubic:

$$\frac{b^3-a^3}{b-a} = a^2+ab+b^2 \overset{?}{=} 3c^2$$

where $c = a+t(b-a)$, $t\in(0,1)$.

$$c^2 = (a^2+ab+b^2)/3 \Rightarrow c = \sqrt{\frac{a^2+ab+b^2}{3}}$$

$$a+t(b-a) = \sqrt{\frac{a^2+ab+b^2}{3}}$$

$$t = \frac{\sqrt{\frac{a^2+ab+b^2}{3}} - a}{b-a}$$

Try $a=1$, $b=i$; $a^2+ab+b^2 = i$; $\sqrt{i} = \pm\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)$

$$t = \frac{\pm\left(\frac{1}{\sqrt{6}}+\frac{1}{\sqrt{6}}i\right)-1}{-1+i}$$

$$= \frac{1}{2}\left(1 \mp \frac{1}{\sqrt{6}} \mp \frac{1}{\sqrt{6}}i\right)(1+i)$$

$$= \frac{1}{2}\left(1 \mp \frac{1}{\sqrt{6}} \pm \frac{1}{\sqrt{6}} + i\left(1 \mp \frac{1}{\sqrt{6}} \mp \frac{1}{\sqrt{6}}\right)\right)$$

$$= \frac{1}{2}\left(1 + i\left(1 \mp \frac{2}{\sqrt{6}}\right)\right) \notin (0,1)$$

Therefore, there is no $c$ on the line segment from $a$ to $b$ such that $f'(c) = a^2+ab+b^2$.

A classic example is also

$$f(z) = e^{z}; \quad a = 0,\ b = 2\pi i$$

$$f'(z) = e^{z}; \quad \frac{f(b) - f(a)}{b-a} = 0$$

however $e^z \neq 0$ for any $z \in \mathbb{C}$.

#2. $f(z) = u(x,y) + i\,v(x,y)$, $z = x+iy$

$v = 0$ in $R \Rightarrow v_x = 0,\ v_y = 0$

Since $v_x = -u_y,\ v_y = u_x$

$\Rightarrow u_x = 0,\ u_y = 0$

at every point $(x,y)$ in $R$

Since $R$ is open, connected

$\Rightarrow u(x,y) = \text{Constant}, \quad (x,y) \in R.$

$\Rightarrow f(z) = \text{constant}, \quad z \in R.$

#3.
$$f_\theta(z) = \lim_{r\to 0} \frac{f(z+re^{i\theta}) - f(z)}{re^{i\theta}}$$

let $f = u + iv$ ; $z = x + iy$

let's express $\dfrac{f(z+dz) - f(z)}{dz}$

where $dz = dx + idy$ in terms of $dz$ and $dz^* = dx - idy$.

We have

$$\frac{f(z+dz)-f(z)}{dz} = \frac{u_x\,dx + u_y\,dy + i(v_x\,dx + v_y\,dy)}{dx+idy} + o(dz)$$

$$= \frac{u_x \frac{dz+dz^*}{2} + u_y \frac{dz-dz^*}{2i} + i\left(v_x \frac{dz+dz^*}{2} + v_y \frac{dz-dz^*}{2i}\right)}{dz} + o(dz)$$

$$= \frac{\left(\frac{1}{2}(u_x+v_y) + \frac{1}{2}i(v_x-u_y)\right)dz + \left(\frac{1}{2}(u_x-v_y) + \frac{1}{2}i(v_x+u_y)\right)dz^*}{dz} + o(dz)$$

$$= \frac{1}{2}(u_x+v_y) + \frac{1}{2}i(v_x-u_y) + \left(\frac{1}{2}(u_x-v_y) + \frac{1}{2}i(v_x+u_y)\right)\frac{dz^*}{dz} + o(dz).$$

$$= \frac{\partial f}{\partial z} + \frac{\partial f}{\partial z^*}\frac{dz^*}{dz} + o(dz),$$

where we used

$$\frac{\partial f}{\partial z} = \frac{1}{2}(\partial_x - i\partial_y)f = \frac{1}{2}(u_x+u_y) + \frac{1}{2}i(v_x-u_y)$$

$$\frac{\partial f}{\partial z^*} = \frac{1}{2}(\partial_x + i\partial_y)f = \frac{1}{2}(u_x-v_y) + \frac{1}{2}i(v_x+u_y)$$

Thus

$$f_\theta(z) = \lim_{r \to 0} \frac{f(z + re^{i\theta}) - f(z)}{re^{i\theta}}$$

$$= \lim_{r \to 0} \left( \frac{\partial f}{\partial z} + \frac{\partial f}{\partial z^*} \frac{re^{-i\theta}}{re^{i\theta}} + o(r) \right)$$

$$= \frac{\partial f}{\partial z} + \frac{\partial f}{\partial z^*} e^{-2i\theta}$$

We see that for $z$ fixed, the quantity $f_\theta(z)$ runs around the circle with center at $\frac{\partial f}{\partial z}$ and radius $\left|\frac{\partial f}{\partial z^*}\right|$ twice clockwise, starting from $\frac{\partial f}{\partial z} + \frac{\partial f}{\partial z^*}$ when $\theta = 0$.

The circle degenerates into a single point $\left(\frac{\partial f}{\partial z}\right)$ when $\frac{\partial f}{\partial z^*} = 0$, i.e. $u_x = v_y$, $v_x = -u_y$ (Cauchy-Riemann equations.)

#4.

(a) $$\frac{d}{dz} a^z = \frac{d}{dz} e^{z\ln(a)} = \frac{d}{dz} e^{z(\ln_p(a)+2\pi i n)}$$

$$= (\ln_p(a)+2\pi i n)\, e^{z(\ln_p(a)+2\pi i n)} = \ln(a)\, a^z,$$

where the same branch (i.e. same $n$-value) is used in both $\ln(a)$ and $a^z$.

(b) $$\frac{d}{dz} z^z = \frac{d}{dz} e^{z\ln(z)} = \frac{d}{dz} e^{z(\ln_p(z)+2\pi i n)}$$

$$= (\ln_p(z)+1+2\pi i n)\, e^{z(\ln_p(z)+2\pi i n)}$$

$$= (1+\ln(z))\, z^z$$

where the same branch (same $n$-value) is used in both $\ln(z)$ and $z^z$.

(c) $$\cosh(z) = w \Rightarrow \frac{e^z+e^{-z}}{2} = w \Rightarrow e^{2z}+1 = 2we^z$$

$$\Rightarrow e^z = w + \underbrace{\sqrt{w^2-1}}_{2 \text{ values}} \Rightarrow z = \ln\left(w+\sqrt{w^2-1}\right)$$

Pick a particular branch of $\sqrt{w^2-1}$, denote $\left(\sqrt{w^2-1}\right)_1$

Then $\left(\sqrt{w^2-1}\right)_2 = -\left(\sqrt{w^2-1}\right)_1$

$$\left(w+\left(\sqrt{w^2-1}\right)_1\right)\left(w-\left(\sqrt{w^2-1}\right)_1\right) = w^2-(w^2-1) = 1$$

$$\ln\left(w-\left(\sqrt{w^2-1}\right)_1\right) = \ln\frac{1}{w+\left(\sqrt{w^2-1}\right)_1} = -\ln\left(w+\left(\sqrt{w^2-1}\right)_1\right)$$

$$\operatorname{arccosh}(z) = \begin{cases} \ln\left(z+\left(\sqrt{z^2-1}\right)_1\right) + 2\pi n i \\ -\ln\left(z+\left(\sqrt{z^2-1}\right)_1\right) + 2\pi n i \end{cases}$$

$$\frac{d}{dz}\ln\left(z+\left(\sqrt{z^2-1}\right)_1\right) = \frac{1+\frac{z}{\left(\sqrt{z^2-1}\right)_1}}{z+\left(\sqrt{z^2-1}\right)_1} = \frac{1}{\left(\sqrt{z^2-1}\right)_1}$$

$$\frac{d}{dz}\operatorname{arccosh}(z) = \begin{cases} \dfrac{1}{\left(\sqrt{z^2-1}\right)_1} \\ -\dfrac{1}{\left(\sqrt{z^2-1}\right)_1} = \dfrac{1}{\left(\sqrt{z^2-1}\right)_2}. \end{cases}$$

Thus

$$\frac{d}{dz}\operatorname{arccosh}(z) = \frac{1}{\sqrt{z^2-1}}$$

where the same branch of $\sqrt{z^2-1}$ is used as in the formula

$$\operatorname{arccosh}(z) = \ln\left(w + \sqrt{z^2-1}\right).$$

#5. $f(z)$ holomorphic on $R$ ; $f(z) \neq 0$ on $R$.
(analytic)

Then
$$\ln|f(z)| + i \arg(f(z)) = \ln f(z)$$
(locally about every $z_0 \in R$; no problem to define single-valued $\arg(f(z))$ then)

$\ln f(z)$ analytic $\Rightarrow$ $\operatorname{Re} \ln f(z) = \ln|f(z)|$ is harmonic.

Computational solution:

Use $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$; $\frac{\partial}{\partial z^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

Then
$$\frac{\partial}{\partial z^*}(u+iv) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u+iv) = \frac{1}{2}\left((u_x - v_y) + i(v_x + u_y)\right)$$
$$= 0 \quad (C\text{-}R)$$
$$\frac{\partial}{\partial z}(u - iv) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u - iv) = \frac{1}{2}\left((u_x - v_y) - i(v_x + u_y)\right) = 0$$

Also
$$\frac{\partial}{\partial z}\frac{\partial}{\partial z^*} = \frac{1}{4}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$
and
$$\frac{\partial}{\partial z^*}\frac{\partial}{\partial z} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

Thus, to show that $\ln|f(z)| = \frac{1}{2}\ln f(z) f^*(z)$ is harmonic, enough to show

$$\frac{\partial}{\partial z}\frac{\partial}{\partial z^*} F = 0, \qquad F(z) = \ln f(z) f^*(z).$$

Compute:
$$\frac{\partial}{\partial z^*} F = \frac{\partial}{\partial z^*} \ln f f^* = \frac{1}{f f^*}\left(\underbrace{\frac{\partial f}{\partial z^*}}_{=0} f^* + f \frac{\partial f^*}{\partial z^*}\right)$$
$$= \frac{1}{f^*}\frac{\partial f^*}{\partial z^*}.$$

$$\frac{\partial}{\partial z}\frac{\partial}{\partial z^*}F = \frac{\partial}{\partial z}\left(\frac{1}{f^*}\frac{\partial f^*}{\partial z^*}\right) =$$

$$= -\frac{1}{f^{*2}}\underbrace{\frac{\partial f^*}{\partial z}}_{=0}\frac{\partial f^*}{\partial z^*} + \frac{1}{f^*}\frac{\partial}{\partial z}\frac{\partial}{\partial z^*}f^*$$

$$= \frac{1}{f^*}\frac{\partial}{\partial z^*}\underbrace{\frac{\partial}{\partial z}f^*}_{=0} = 0.$$

*Now suppose, more generally, that*

$$F(x,y) = g(ff^*),$$

(so contour lines of $F$ coincide with contour lines of $|f|$)

and $f$ is holomorphic (analytic.)

Then $$\frac{\partial}{\partial z^*}F = g'(ff^*)\left(\overbrace{\frac{\partial f}{\partial z^*}}^{=0}f^* + f\frac{\partial f^*}{\partial z^*}\right)$$

$$= g'(ff^*)\, f\,\frac{\partial f^*}{\partial z^*},$$

$$\frac{\partial}{\partial z}\frac{\partial}{\partial z^*}F = g''(ff^*)\left|f\frac{\partial f^*}{\partial z^*}\right|^2 + g'(ff^*)\overbrace{\frac{\partial f}{\partial z^*}}^{=0}\frac{\partial f^*}{\partial z^*}$$

$$+ g'(ff^*)\, f\underbrace{\frac{\partial^2 f^*}{\partial z^*\partial z}}_{=0}$$

Thus, $$\frac{\dfrac{\partial^2 F}{\partial z\,\partial z^*}}{\left|\dfrac{\partial F}{\partial z^*}\right|^2} = \frac{g''(ff^*)\cancel{\left|f\frac{\partial f^*}{\partial z^*}\right|^2}}{(g'(ff^*))^2\cancel{\left|f\frac{\partial f^*}{\partial z^*}\right|^2}} = G(ff^*)$$

$$= h(F).$$

since
$$\frac{\partial}{\partial z} g'(ff^*) = g''(ff^*) \frac{\partial}{\partial z}(ff^*)$$
$$= g''(ff^*)\left(\frac{\partial f}{\partial z} f^* + f \underbrace{\frac{\partial f^*}{\partial z}}_{=0}\right) = g''(ff^*)\, f^* \left(\frac{\partial f^*}{\partial z^*}\right)^*$$

and
$$\frac{\partial f^*}{\partial z^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u - iv) = \frac{1}{2}\left(u_x + v_y - i\left(v_x - u_y\right)\right)$$
$$= \left(\frac{\partial f}{\partial z}\right)^*.$$

#6.

$$\frac{\partial f}{\partial z^*} = \frac{1}{2}(\partial_x + i\partial_y)(u+iv) = \frac{1}{2}(u_x - v_y) + \frac{1}{2}i(v_x + u_y) = 0$$

$$\iff u_x = v_y, \quad v_x = -u_y \qquad \text{(Cauchy-Riemann)}$$

Notice that if $f'(z)$ exists, then

$$f'(z) = u_x + i v_x = v_y - i u_y$$
$$= \frac{1}{2}(u_x + v_y) + \frac{1}{2}i(v_x - u_y)$$

and $\frac{\partial f}{\partial z} = \frac{1}{2}(\partial_x - i\partial_y)(u+iv) = \frac{1}{2}(u_x + v_y) + \frac{1}{2}i(v_x - u_y)$

so $f'(z) = \frac{\partial f}{\partial z}$.

Conversely, if $u_x$, $u_y$, $v_x$, $v_y$ are continuous, then

$$u(x+h, y+k) - u(x,y) = u(x+h, y+k) - u(x, y+k)$$
$$+ u(x, y+k) - u(x,y) = u_x(x_h, y+k)h + u_y(x, y_k)k$$
$$= u_x(x,y)h + u_y(x,y)k + \varepsilon_1(h,k)h + \varepsilon_2(h,k)k$$

where $\varepsilon_1(h,k) = u_x(x_h, y+k) - u_x(x,y) \xrightarrow[(h,k)\to 0]{} 0$

and $\varepsilon_2(h,k) = u_y(x, y_k) - u_y(x,y) \xrightarrow[(h,k)\to 0]{} 0$.

Similarly

$$v(x+h, y+k) - v(x,y) = v_x(x,y)h + v_y(x,y)k + \varepsilon_3(h,k)h + \varepsilon_4(h,k)k,$$

where $\varepsilon_{3,4}(h,k) \longrightarrow 0$ as $(h,k) \to 0$.

In complex notation,

$$f(z+dz) - f(z) = u(x+h, y+k) - u(x,y) + i\,(v(x+h, y+k) - v(x,y))$$

$$\left[\begin{array}{l} z = x+iy \\ dz = h+ik \end{array}\right]$$

$$= u_x h + u_y k + i(v_x h + v_y k) + o(dz),$$

[where $o(dz) = \varepsilon_1(h,k)h + \varepsilon_2(h,k)k + i(\varepsilon_3(h,k)h + \varepsilon_4(h,k)k)$]

$$= u_x \frac{dz+dz^*}{2} + u_y \frac{dz-dz^*}{2i} + i\left(v_x \frac{dz+dz^*}{2} + v_y \frac{dz-dz^*}{2i}\right) + o(dz).$$

$$= \left(\frac{1}{2}(u_x+u_y) + \frac{1}{2}i(v_x-u_y)\right)dz + \left(\frac{1}{2}(u_x-v_y)+\frac{1}{2}i(v_x+u_y)\right)dz^* + o(dz)$$

$$= \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial z^*}dz^* + o(dz),$$

$$= \frac{\partial f}{\partial z}dz + o(dz), \quad \text{since by C–R } \frac{\partial f}{\partial z^*} = 0.$$

Thus,

$$\frac{f(z+dz) - f(z)}{dz} = \frac{\partial f}{\partial z} + o(1)$$

$$\Rightarrow \lim_{dz\to 0} \frac{f(z+dz)-f(z)}{dz} = \frac{\partial f}{\partial z}.$$

#7. $f(z) = u(x,y) + i\,v(x,y)$ ; $z = x+iy$

$C$ – simple closed contour;

$A$ = Interior of $C$.

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy)$$

$$= \int_C u\,dx - v\,dy + i\,(v\,dx + u\,dy)$$

$$= \iint_A (-v_x - u_y) + i\,(u_x - v_y)\,dA$$

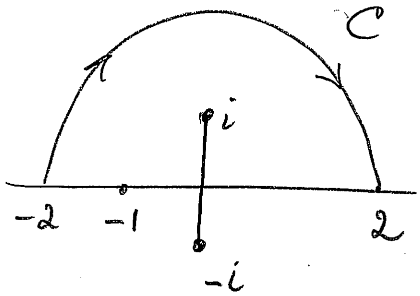
$$= 2i \iint_A \tfrac{1}{2}(u_x - v_y) + \tfrac{1}{2}(v_x + u_y)\,dA$$

$$= 2i \iint_A \frac{\partial f}{\partial z^*}\,dA$$

since $\dfrac{\partial f}{\partial z^*} = \tfrac{1}{2}(\partial_x + i\partial_y)(u+iv) = \tfrac{1}{2}(u_x - v_y) + \tfrac{1}{2}i(v_x + u_y)$.

\#8.

Find $\displaystyle\int_C \frac{dz}{\left(\sqrt{1+z^2}\right)_1}$




where $\left(\sqrt{1+z^2}\right)_1$ indicates the branch of $\sqrt{1+z^2}$ continuous on $\mathbb{C}\setminus i[-1,1]$ and such that $\left(\sqrt{1+z^2}\right)_1\Big|_{z=-1} = -\sqrt{2}$, and $C$ is as pictured.

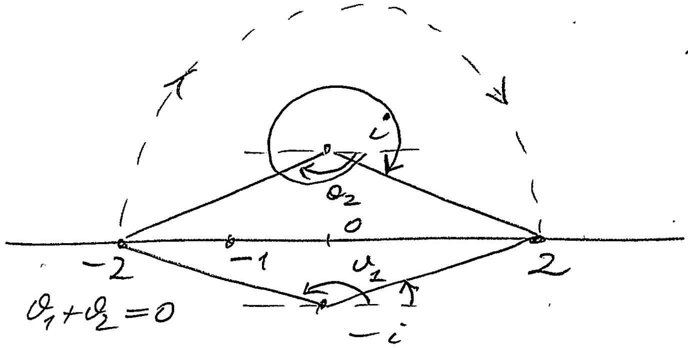
Solution: Let $(\operatorname{arcsinh}(z)) = \ln\left(z+\left(\sqrt{z^2+1}\right)_1\right)$ then (as in problem #4(c))

$$\frac{d}{dz}\ln\left(z+\left(\sqrt{z^2+1}\right)\right) = \frac{1}{\left(\sqrt{1+z^2}\right)_1}, \text{ so}$$

$$\int_C \frac{dz}{\left(\sqrt{1+z^2}\right)_1} = \left[\ln\left(z+\left(\sqrt{z^2+1}\right)_1\right)\right]_{z=-2}^{z=2}$$

The key step is to construct the branch of $\sqrt{1+z^2}$ carefully: particularly we want to make sure that the said branch is continuous on $C$.

We have $\sqrt{1+z^2} = \sqrt{(z+i)(z-i)} = (\rho_1\rho_2)^{\frac{1}{2}} e^{\frac{1}{2}i(\vartheta_1+\vartheta_2)} e^{i\pi n}$ $(n=0,1)$



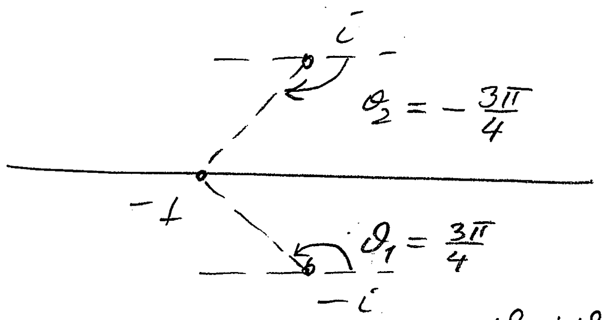

$z=-2$: $\vartheta_1+\vartheta_2=0$

on $C$: $\vartheta_1$ decreases from a value in II quadr to a value in Ist quadr

$\vartheta_2$ decreases from a value in III quadr to a value in IV quadr.

$z=2$: $\vartheta_1+\vartheta_2 = -2\pi$

at $z=-1$:




$\vartheta_1 + \vartheta_2 = 0$

$$\left(\sqrt{1+z^2}\right)_1\Big|_{z=-1} = \left(\sqrt{2}\cdot\sqrt{2}\right)^{\frac{1}{2}}\cdot e^{i\pi n} = -\sqrt{2}$$

$\Rightarrow n = 1$

at $z = 2$:

$\frac{1}{2}(\vartheta_1+\vartheta_2) = -\pi$

$$\left(\sqrt{1+z^2}\right)_1\Big|_{z=2} = \left(\sqrt{5}\sqrt{5}\right)^{\frac{1}{2}}\cdot e^{-i\pi} e^{i\pi} = \sqrt{5}$$

at $z=-2$:

$\frac{1}{2}(\vartheta_1+\vartheta_2) = 0$

$$\left(\sqrt{1+z^2}\right)_1\Big|_{z=-2} = \left(\sqrt{5}\sqrt{5}\right)^{\frac{1}{2}}\cdot e^{i\pi} = -\sqrt{5}$$

at $z = 2i$

$\frac{1}{2}(\vartheta_1+\vartheta_2) = -\pi/2$

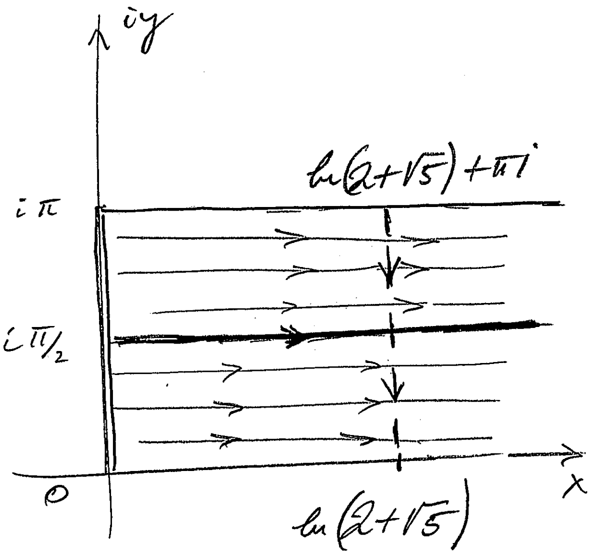
$$\left(\sqrt{1+z^2}\right)_1\Big|_{z=2i} = (2\cdot 1)^{1/2} e^{-i\frac{\pi}{2}} e^{i\pi} = i\sqrt{2}.$$

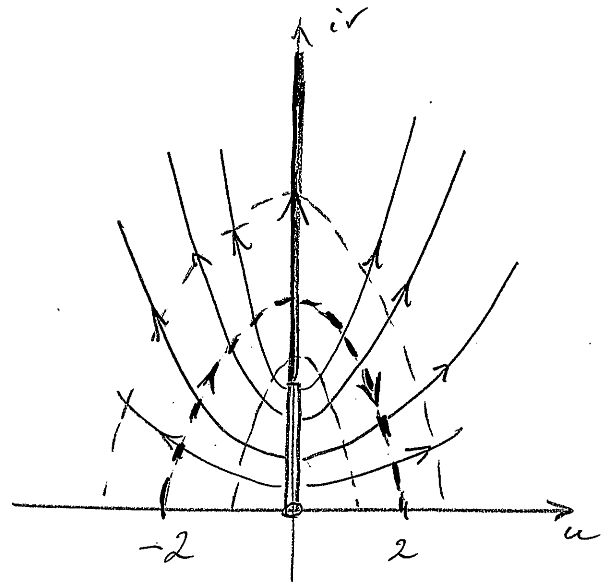
Thus,

$$\int_C \frac{dz}{\left(\sqrt{1+z^2}\right)_1} = \ln\left(2+\sqrt{5}\right) - \ln\left(-2-\sqrt{5}\right)$$
$$= \ln\left(2+\sqrt{5}\right) - \ln\left(2+\sqrt{5}\right) - \pi i$$
$$= -\pi i$$

(since the intermediate value at $z=2i$ corresponds to $\arg\left(z+\left(\sqrt{z^2+1}\right)_1\right) = \pi/2$ we used $\ln\left(-2-\sqrt{5}\right) = \ln\left(2+\sqrt{5}\right)+\pi i$)

Illustration: $\sinh(x+iy) = \sinh(x)\cos(y) + i\cosh(x)\sin(y)$




$y = \frac{\pi}{2} \Rightarrow \sin(y) = 1$
$\cos(y) = 0$
(part of pos. imag. axis)

$y = 0 \Rightarrow \sin(y) = 0$
$\cos(y) = 1$
(pos. real axis)

$y = \pi \Rightarrow \sin(y) = 0$
$\cos(y) = -1$
(neg. real axis)

Integral over half-ellipse is the same as the integral over half-circle.