Homework 5

Due on Wed. Feb 27, 2019.

1. Show, by giving a counterexample, that the following analog of the Mean-Value Theorem is false for analytic functions of a complex variable: *Suppose f is analytic in an open set W that contains the line segment ℓ connecting two values a and b in $\mathbb{C}$. Then there exist a value $c \in \ell$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

2. (Problem 2-1:1) Show that no purely real function defined on a region (*i. e.* open, connected set) R in the complex plane can be analytic, unless it is constant.

3. (Problem 2-1:4) Even if a given $f(z)$ is not analytic, the limit

$$f_\theta(z) = \lim_{r \to 0} \frac{f(z + re^{i\theta}) - f(z)}{re^{i\theta}}$$

may still exist for each fixed choice of θ. Show that the set $\{f_\theta(z) : \theta \in (-\pi, \pi]\}$ lies on a circle when plotted in the complex plane. Obtain the center and the radius of the circle. When does the circle degenerate into a single point?

4. (Problem 2-1:5) If a is a complex constant, differentiate a^z, z^z and $\operatorname{arccosh}(z)$. Explain how to determine the branch of the result.

5. (Problem 2-1:6) Prove that $\ln |f|$ is harmonic if f is analytic. Show that a necessary condition that the curves $F(x, y) = c$, c a real parameter, coincide with the contour lines of the modulus of some analytic function is that

$$\frac{F_{xx} + F_{yy}}{F_x^2 + F_y^2}$$

be some function of F.

6. Suppose $u(x, y)$ and $v(x, y)$ are two well-behaved functions (*i. e.* they possess continuous partial derivatives in a region R in $\mathbb{R}^2$). We write $z = x + iy$, $z^* = x - iy$, and $f = u + iv$, also $x = \frac{z+z^*}{2}$, $y = \frac{z-z^*}{2i}$, so

$$f(x, y) = f\Big(\frac{z + z^*}{2}, \frac{z - z^*}{2i}\Big).$$

We then introduce formal partial derivatives with respect to z and z^*:

$$\frac{\partial f}{\partial z} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\Big), \qquad \frac{\partial f}{\partial z^*} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big).$$

Show that the Cauchy-Riemann equations hold if and only if $\partial f/\partial z^* = 0$. Also show that in that case $F(x+iy) = f(x,y)$ is differentiable with respect to z in the sense of complex-valued derivative, and $F'(z) = \partial f/\partial z$.

7. (Problem 2-2:4) For $F(x+iy) = f(x,y)$ as in the previous problem (not necessarily an analytic function), show that

$$\int_C F(z)\,dz = 2i \iint_A \frac{\partial f}{\partial z^*}\,dx\,dy$$

for any closed contour C containing an area A within the region R.

8. (Problem 2-2:5) Evaluate

$$\int \frac{dz}{(1+z^2)^{1/2}}$$

from $z = -2$ to $z = 2$ around the semicircle in the upper half-plane satisfying $|z| = 2$. We complete the definition of the function $(1+z^2)^{1/2}$ by choosing as branch line that part of the imaginary axis satisfying $|y| \le 1$; also at $z = -1$, $(1+z^2)^{1/2}$ is chosen as the negative squate root of 2.