MATH 655 HOMEWORK 4 – SOLUTIONS

#1. Prove that $z \longmapsto z^*$ is not fractional linear.

Suppose $\bar{z} = \dfrac{az+b}{cz+d}$

$z=0 \Rightarrow \frac{b}{d} = 0 \Rightarrow b = 0$

$z = \infty \Rightarrow \frac{a}{c} = \infty \Rightarrow c = 0, a \neq 0$

$z = 1 \Rightarrow \frac{a}{d} = 1 \Rightarrow z \equiv z^*$
false for $\operatorname{Im}(z) \neq 0$.

#2. $w = \dfrac{az+b}{cz+d}$ is such that $w \in \mathbb{R}$ whenever $z \in \mathbb{R}$.

$z \longmapsto w$ is invertible (otherwise $z \mapsto w$ is constant, statement triv. true.)

WLOG, $a \in \mathbb{R}$, otherwise multiply both numer. and den. by a^*.

$a = 0 \Rightarrow \frac{1}{w} = \frac{c}{b} z + \frac{d}{b}$

$z = 0 \Rightarrow \frac{d}{b} \in \mathbb{R}$;

$z = z_0, \text{ so } \frac{c}{b} z_0 + \frac{d}{b} \neq 0 \Rightarrow \frac{c}{b} \in \mathbb{R}$.
$\in \mathbb{R}$

$a \neq 0 \Rightarrow w = 0$ for $z = -\frac{b}{a} \in \mathbb{R}$
$\Rightarrow b \in \mathbb{R}$.

$\lim\limits_{\substack{z \to \infty \\ z \in \mathbb{R}}} \dfrac{az+b}{cz+d} = \dfrac{a}{c}$ (or ∞ if $c=0$) $\Rightarrow c \in \mathbb{R}$

$z \neq -\frac{d}{c} \Rightarrow cz + d = \dfrac{az+b}{w} \in \mathbb{R}$ $\left(\substack{z, w \\ \in \mathbb{R}}\right)$
$\Rightarrow d \in \mathbb{R}$.

#3. $z_1, z_2, z_3, z_4 \in \mathbb{C} \longmapsto 1, -1, k, -k$

$$(1, -1, k, -k) = (z_1, z_2, z_3, z_4)$$

$$\frac{1-k}{1+k} : \frac{-1-k}{-1+k} = \rho := (z_1, z_2, z_3, z_4)$$

$$\left(\frac{1-k}{1+k}\right)^2 = \rho \Rightarrow \frac{1-k}{1+k} = \pm\sqrt{\rho}$$

$$k = \frac{1 \mp \sqrt{\rho}}{1 \pm \sqrt{\rho}}$$; the two values are reciprocals of each other.

Let $S(z) = (z, -1, k, -k)$, then the four points z_1, z_2, z_3, z_4 are mapped to $1, -1, k, -k$.

#4. $z \longmapsto z_s$ reflection through circle passing through z_1, z_2, z_3

$$(z_s, z_1, z_2, z_3) = (z, z_1, z_2, z_3)^*$$ (complex conjugate)

$$\Rightarrow (z_s^*, z_1^*, z_2^*, z_3^*) = (z, z_1, z_2, z_3)$$

$z_s \longmapsto z_{ss}$ reflection through circle passing through z_4, z_5, z_6

$$(z_{ss}, z_4, z_5, z_6) = (z_s, z_4, z_5, z_6)^*$$

$$= (z_s^*, z_4^*, z_5^*, z_6^*)$$

$\Rightarrow$ $z \longmapsto z_s^*$ is fr. linear

$z_s^* \longmapsto z_{ss}$ is fr. linear.

The composition is fr. linear.

#5. $C = \{ z \in \mathbb{C} : |z - a| = R \}$.

$$(z, z_1, z_2, z_3) = \frac{z - z_2}{z - z_3} \Big/ \frac{z_1 - z_2}{z_1 - z_3} = \frac{(z-a) - (z_2 - a)}{(z-a) - (z_3 - a)} \Big/ \frac{(z_1 - a) - (z_2 - a)}{(z_1 - a) - (z_3 - a)}$$

$$= (z - a, z_1 - a, z_2 - a, z_3 - a).$$

Since $|z_i - a|^2 = (z_i - a)(z_i - a)^* = R^2$

$$(z_i - a)^* = \frac{R^2}{z_i - a}$$

Also $(z_1, z_2, z_3, z_4)^* = (z_1^*, z_2^*, z_3^*, z_4^*)$

$$\Rightarrow (z_1, z_2, z_3, z_4)^* = \left(z^* - a^*, \frac{R^2}{z_1 - a}, \frac{R^2}{z_2 - a}, \frac{R^2}{z_3 - a} \right).$$

$$\left(w, \frac{R^2}{w_1}, \frac{R^2}{w_2}, \frac{R^2}{w_3} \right) = \frac{w - \frac{R^2}{w_2}}{w - \frac{R^2}{w_3}} \Bigg/ \frac{\frac{R^2}{w_1} - \frac{R^2}{w_2}}{\frac{R^2}{w_1} - \frac{R^2}{w_3}}$$

$$= \frac{w_2 - \frac{R^2}{w}}{w_3 - \frac{R^2}{w}} \Bigg/ \frac{w_2 - w_1}{w_3 - w_1} = \frac{\frac{R^2}{w} - w_2}{\frac{R^2}{w} - w_3} \Bigg/ \frac{w_1 - w_2}{w_1 - w_3}$$

$$= \left(\frac{R^2}{w}, w_1, w_2, w_3 \right) = \left(\frac{R^2}{z^* - a^*}, z_1 - a, z_2 - a, z_3 - a \right)$$

$$= \left(\frac{R^2}{z^* - a} + a, z_1, z_2, z_3 \right)$$

Thus $$z_s = a + \frac{R^2}{z^* - a}$$

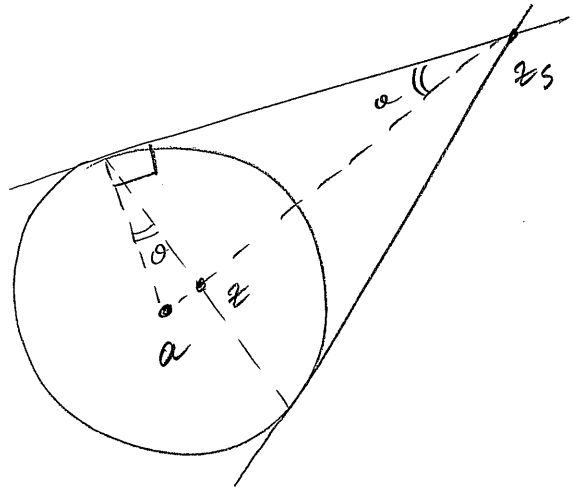
$$z_s - a = \frac{R^2}{z^* - a}$$

$$(z_s - a)(z^* - a^*) = R^2.$$

$$\Rightarrow |z_s - a| |z^* - a^*| = |z_s - a| |z - a| = R$$

$$z_s - a \frac{1}{z - a} = R^2 \frac{1}{|z - a|^2}$$

$$\Rightarrow \frac{z_s - a}{z - a} > 0 \Rightarrow \arg(z_s - a) = \arg(z - a).$$



Verify

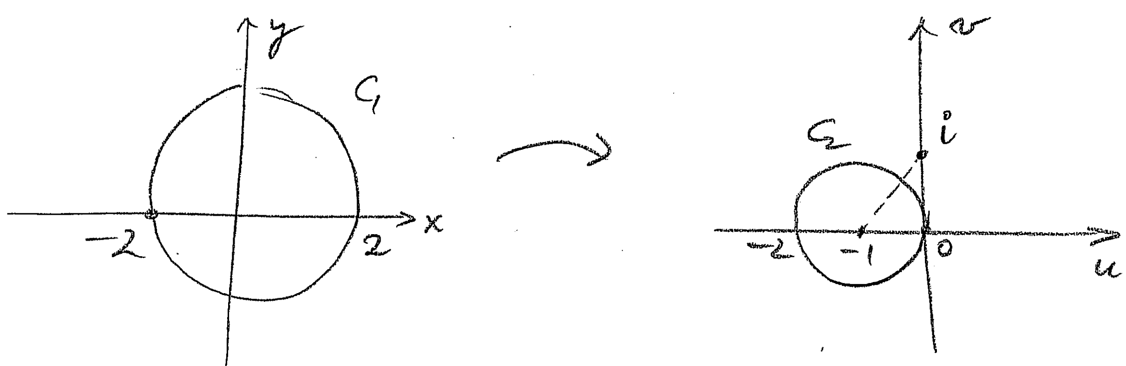
$$|a-z_s| = \frac{R^2}{|a-z|}.$$

$$|a-z| = R\sin\theta$$

$$|a-z_s| = \frac{R}{\sin\theta}$$

$$\Rightarrow \quad |a-z_s| = \frac{R^2}{|a-z|}.$$

#6.



$$-2 \to 0$$
$$0 \to i$$

$$\underset{(C_1)}{(0)_s} = \infty \ ; \qquad \underset{(C_2)}{(i)_s} = -\tfrac{1}{2} + \tfrac{1}{2}i$$

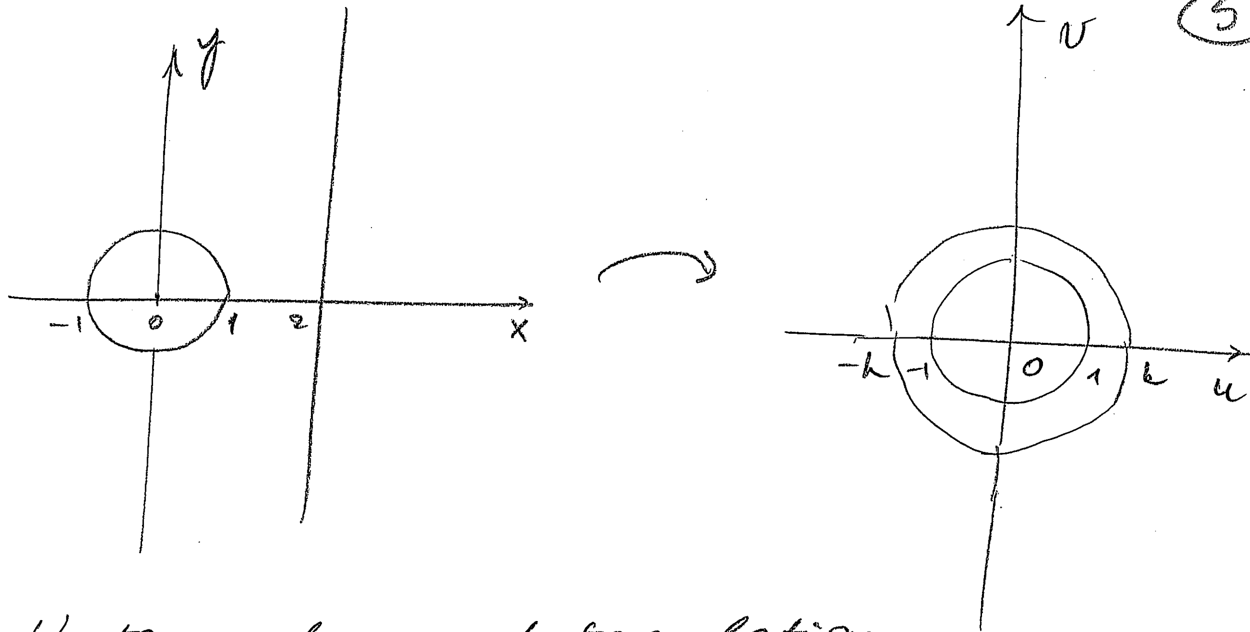
By symmetry, $\infty \longmapsto -\frac{1}{2}+\frac{1}{2}i$

Three points $\Rightarrow$. $\left(w, 0, i, -\frac{1}{2}+\frac{1}{2}i\right) = \left(z, -2, 0, \infty\right)$

$$\frac{w-i}{w+\frac{1}{2}-\frac{1}{2}i} \Big/ \frac{0-i}{0+\frac{1}{2}-\frac{1}{2}i} = \frac{z}{-2}$$

$$\Rightarrow \quad w = \frac{z+2}{-2i-(1+i)z}$$

#7:



Up to scaling and translation,
the transformation can be arranged
to transform the unit circle onto
unit circle.

Choose one that transforms $2, \infty$ onto $k, -k$.

$$(1, -1, k, -k) = (1, -1, 2, \infty)$$

$$\left(\frac{1-k}{1+k}\right)^2 = \frac{1}{3} \Rightarrow k = \frac{\sqrt{3}+1}{\sqrt{3}-1} \text{ or } \frac{\sqrt{3}-1}{\sqrt{3}+1}.$$

Transformation: $(w, -1, k, -k) = (z, -1, 2, \infty)$

$$\frac{w-k}{w+k} \quad \frac{1-k}{1+k} = \frac{z-2}{-3}$$

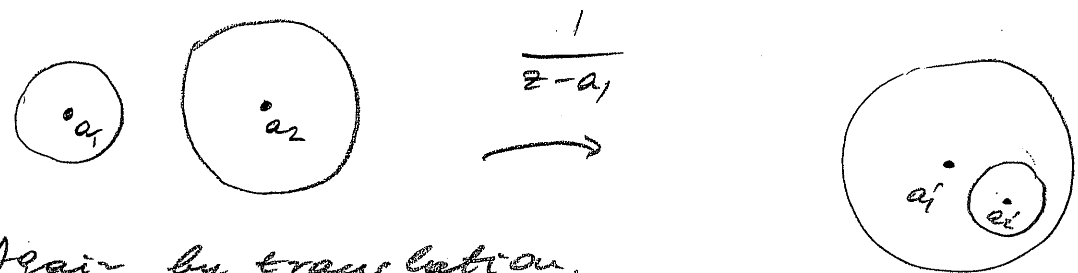
$$\frac{w-k}{w+k} = \xi \Rightarrow w = \frac{k(1+\xi)}{1-\xi} \quad ; \quad \xi = \left(\frac{1+k}{1-k}\right)\frac{2-z}{3}$$

$$w = \frac{k\left(1+\left(\frac{1+k}{1-k}\right)\frac{2-z}{3}\right)}{1-\left(\frac{1+k}{1-k}\right)\frac{2-z}{3}}$$

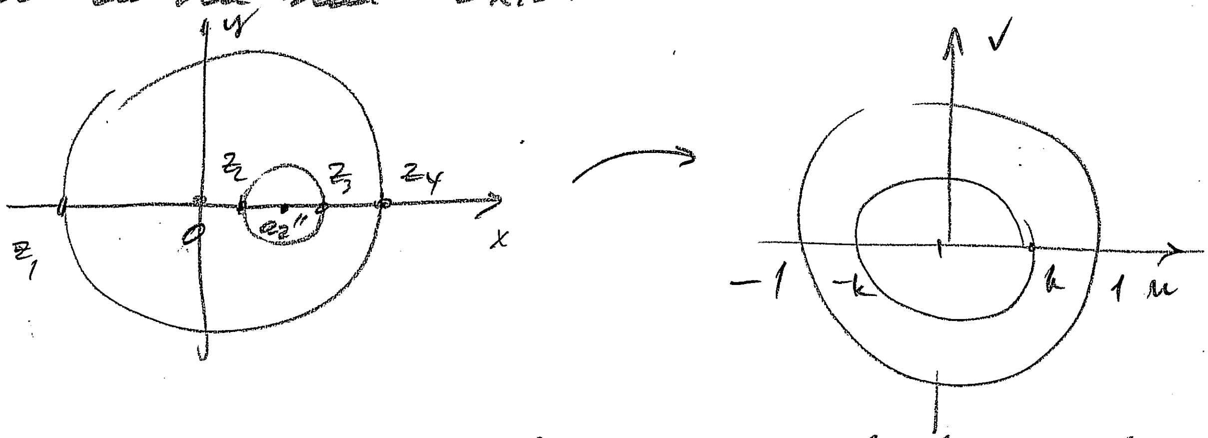
For instance, $w = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} \dfrac{1+\frac{1}{\sqrt{3}}\frac{2-z}{3}}{1-\frac{1}{\sqrt{3}}\frac{2-z}{3}} = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} \dfrac{3\sqrt{3}-2+z}{3\sqrt{3}+2-z}$

transforms the line into the circle of radius $\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$.

#8. If the disks interior to the circles are non-overlapping, can use translation, followed by inversion so that the new circles are "nested":



Again by translation, followed by multiplication by $(a_1'-a_2')^{-1}$ (if non-zero) the circles can be transformed so that both new centers are on the real axis:

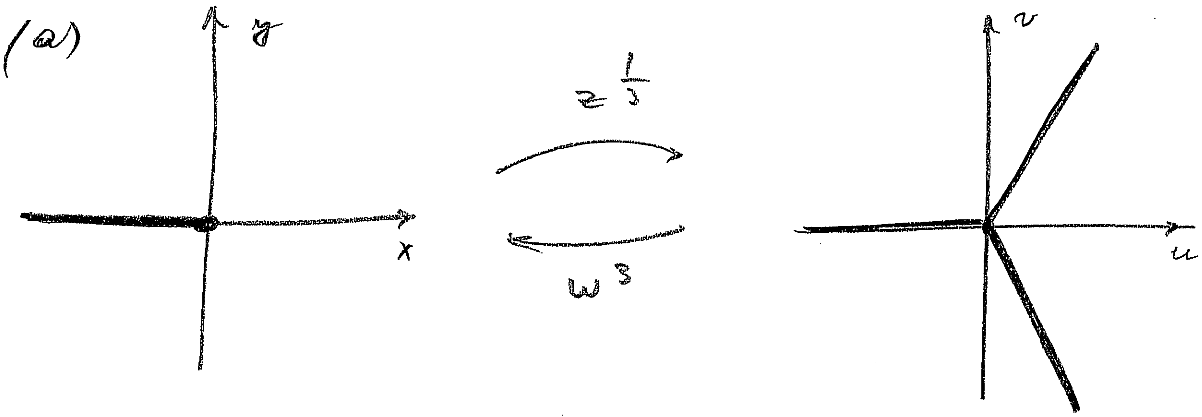


Then use the transformation that maps

$$z_1, z_2, z_3, z_4 \longmapsto -1, -k, k, 1.$$

The symmetry about the real axis is preserved, so the result is always two concentric circles.

#9. (a)



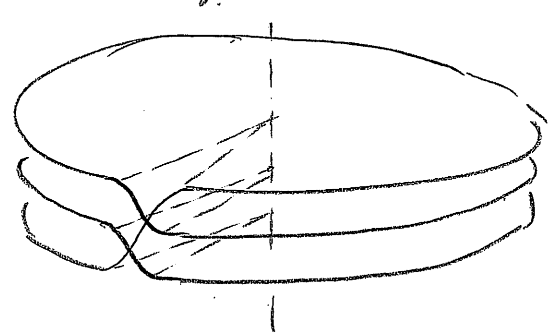
The three branches of $w = z^{\frac{1}{3}}$ correspond to partitioning $\mathbb{C}$ into three sectors and establishing 1-to-1 correspondence between the sectors, and the z-plane with a cut (Example shown)

0 and ∞ are branch points since any closed loop about either results in change of argument by 2π, which corresponds to transition from one branch to another. In the illustrated example,

$$w_i = |z|^{1/3} e^{i\frac{1}{3}\arg_p(z)} e^{i\frac{2\pi n}{3}} \quad , n = 0, 1, 2$$

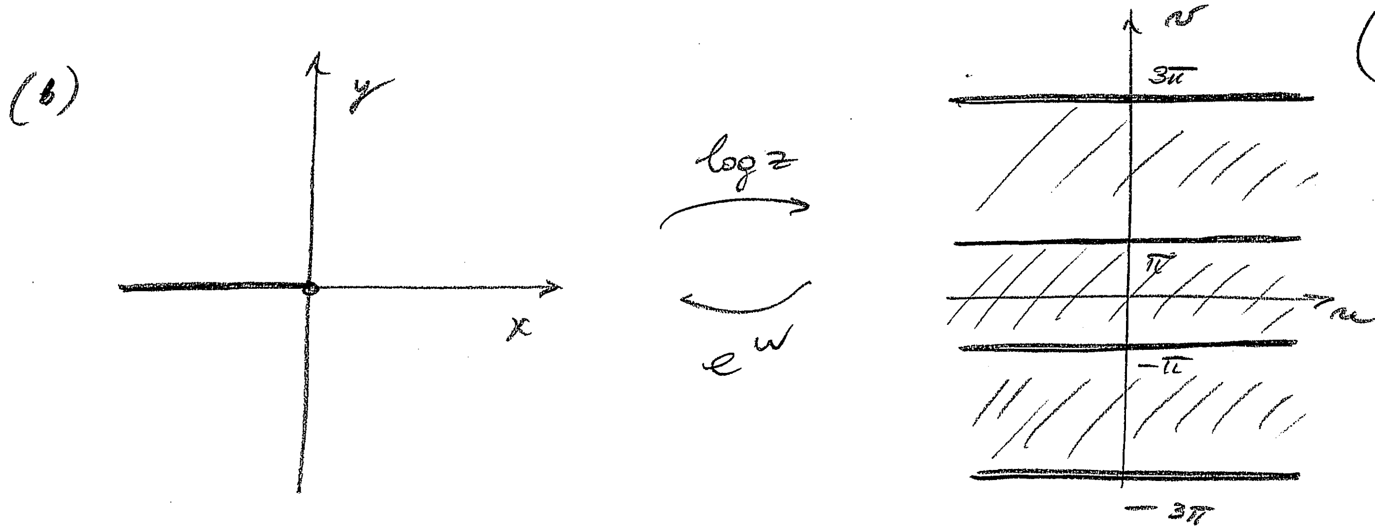
where $\arg_p(z) \in (-\pi, \pi]$. (A different interval would result in a different splitting into branches, and a differently positioned cut.)

Riemann surface: (cut at $\theta = \pm\pi$)



Each copy of the cplx. plane corresponds to a single branch.

(b)



In the inverse map $z=e^w$ each shaded region covers entire $\mathbb{C}$ except the cut.

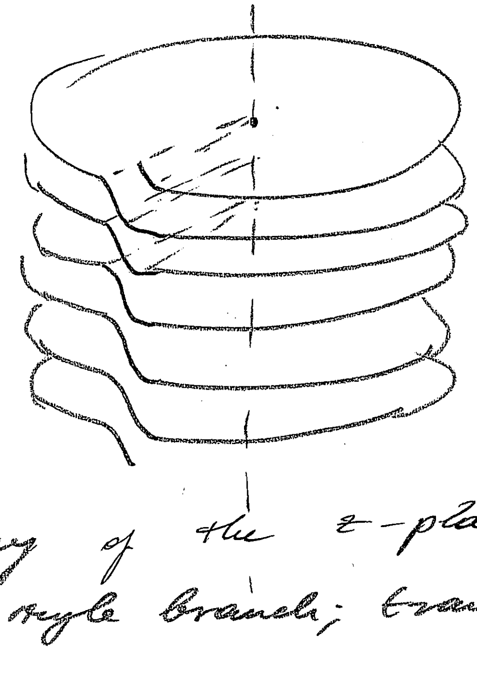
There is a countable family of branches, corresponding to this partition:

$$w_i = \log|z| + i\arg_p(z) + 2\pi i n, \quad n=0,\pm1,\ldots$$

where $\arg_p(z) \in (-\pi,\pi]$

Branch points at $0,\infty$, in the same manner as for $w=z^{\frac{1}{3}}$.

Riemann surface consists of countably many sheets:

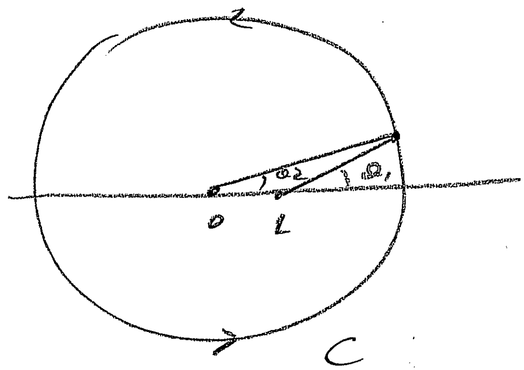


Each copy of the z-plane corresponds to a single branch; transition through the cut.

(c) $w = \sqrt{z(z-1)}$; $w_i = \sqrt{r_1 r_2}\, e^{i\frac{1}{2}(\theta_1+\theta_2)} e^{i\pi n}$

$n = 0, 1$

∞ is not a branch point:



for any circle containing 0 and 1

$\Delta_C(\theta_1+\theta_2) = 4\pi$

$\Rightarrow \Delta_C \frac{1}{2}(\theta_1+\theta_2) = 2\pi$

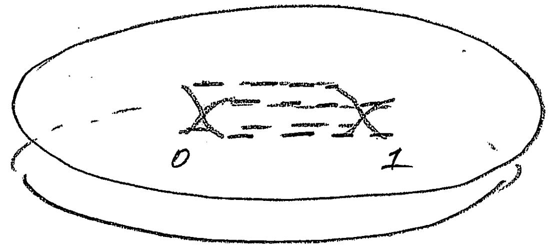
and there is no change in w_i.

0, 1 – branch points

$\Delta_C \frac{1}{2}(\theta_1+\theta_2) = \pi$ in both cases.

Branch cut along line segment connecting 0 and 1 prevents closed loops from encircling both 0 and 1.

Riemann surface:



The two sheets are joined at the cut through [0,1] and remain separated as $z \to \infty$.

#10.

$$5+\sqrt{\frac{z+1}{z-1}} = 5+\sqrt{\frac{r_1}{r_2}}\, e^{i\frac{1}{2}(\theta_1-\theta_2)}\, e^{i\pi n}, \quad n=0,1.$$

$$z+1 = r_1 e^{i\theta_1}; \quad z-1 = r_2 e^{i\theta_2}.$$

$\Delta_C \frac{1}{2}(\theta_1-\theta_2) = 0$ for any circle C containing both -1 and 1

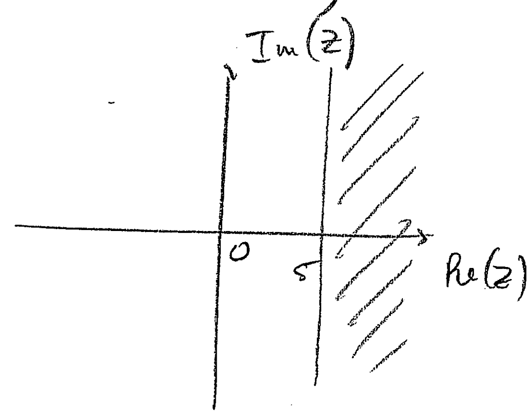
$\Rightarrow$ ∞ is not a branch point.

$z=1$, $z=-1$ are branch points as in #9;

$\arg(z+1) - \arg(z-1) \in (-\pi, \pi)$ on $\mathbb{C} \setminus [-1,1]$

$\Rightarrow$ by choosing branch $n=0$,

$Z = 5+\sqrt{\frac{z+1}{z-1}}$ is as follows:



$\ln Z$ is single-valued on $\{Z : \operatorname{Re} Z > 5\}$

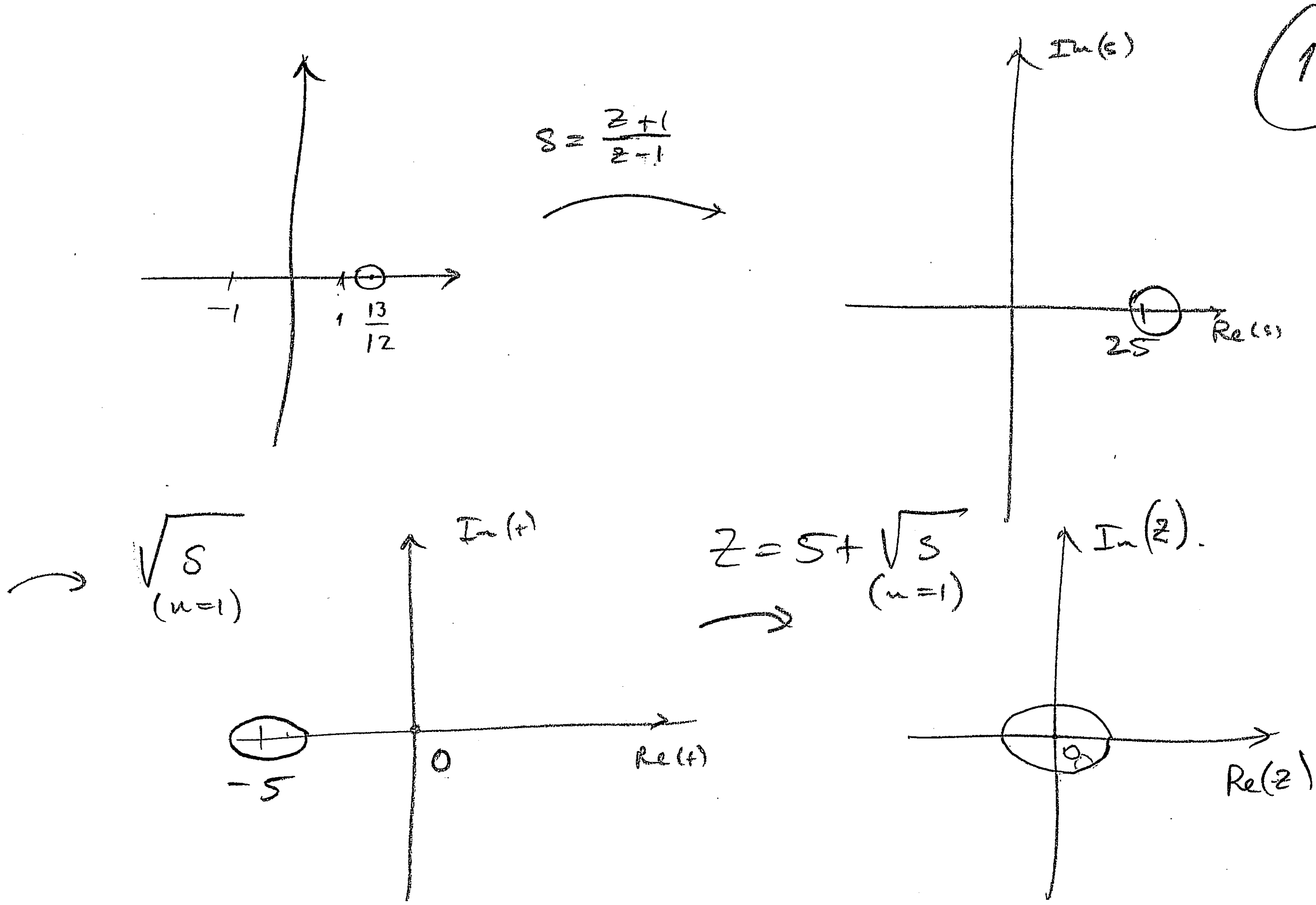
$\Rightarrow$ $f(z) = \ln\left(5+\sqrt{\frac{z+1}{z-1}}\right)$ is single-valued on $\mathbb{C}\setminus[-1,1]$.

However, branch $n=1$ is undefined when $\sqrt{\frac{z+1}{z-1}} = -5$

i.e. $\frac{z+1}{z-1} = 25$, $24z = 26 \Rightarrow z = \frac{13}{12}$.

When z is on a contour encircling $z = \frac{13}{12}$, but not -1 or 1,

and we are using $n=1$ for branch of square root,



Thus, the image of a circle about $z=\frac{13}{12}$ is a loop encircling $Z=0$, where

$$Z = 5+\underset{(n=1)}{\sqrt{\frac{z+1}{z-1}}}.$$

Since $\ln(Z) = \ln|Z| + i\arg(Z) + 2\pi i n$ has $Z=0$ as branch point,

$$\therefore \quad \Delta_C \ln\Big(5+\underset{(n=1)}{\sqrt{\frac{z+1}{z-1}}}\Big) \neq 0$$

and so $z=\frac{13}{12}$ is a branch point for $f(z)=\ln\Big(5+\underset{(n=1)}{\sqrt{\frac{z+1}{z-1}}}\Big)$