MATH 655 Solutions to Homework 3

#1. (a) $\sin z = z - \frac{z^3}{6} + \frac{z^5}{120} - \dots = c_0 + c_1 z + \dots$

$\cos z = 1 - \frac{z^2}{2} + \frac{z^4}{24} - \dots = b_0 + b_1 z + \dots$

$\tan z = \frac{\sin z}{\cos z} = a_0 + a_1 z + \dots$

$(a_0 + a_1 z + \dots)(b_0 + b_1 z + \dots) = c_0 + c_1 z + \dots$

[Solve for $a_0, a_1, \dots$]

$a_0 b_0 = c_0$

$a_0 b_1 + a_1 b_0 = c_1$

$\vdots$

$a_0 b_k + \dots + a_k b_0 = c_k$

$\Rightarrow$

$a_0 = \frac{c_0}{b_0}$

$a_1 = \frac{c_1 - a_0 b_1}{b_0}$

$\vdots$

$a_k = \frac{c_k - a_0 b_k - \dots - a_{k-1} b_1}{b_0}$

$(b_0 = 1)$

$\Rightarrow a_0 = 0; \quad a_1 = 1; \quad a_2 = 0$

$a_3 = c_3 - \cancel{a_0} b_3 - a_1 b_2 - \cancel{a_2} b_1 = -\frac{1}{6} + \frac{1}{2} = \frac{1}{3}$

$a_4 = 0; \quad a_5 = c_5 - \cancel{a_0} b_5 - a_1 b_4 - \cancel{a_2} b_3 - a_3 b_2 - \cancel{a_4} b_1$

$= \frac{1}{120} - \frac{1}{24} - \frac{1}{3}\left(-\frac{1}{2}\right) = \frac{2}{15}$

$a_6 = 0; \quad a_7 = c_7 - \cancel{a_0} b_7 - a_1 b_6 - \cancel{a_2} b_5 - a_3 b_4 - \cancel{a_4} b_3 - a_5 b_2 - \cancel{a_6} b_1$

$= -\frac{1}{5040} + \frac{1}{720} - \frac{1}{3}\frac{1}{24} - \frac{2}{15}\left(-\frac{1}{2}\right)$

$= \frac{-1 + 7 - 70 + 336}{5040} = \frac{17}{315}.$

$\tan z = z + \frac{1}{3} z^3 + \frac{2}{15} z^5 + \frac{17}{315} z^7 + \dots$

(b) $$\tanh(z) = \frac{\sinh(z)}{\cosh(z)} = \frac{(e^{z} - e^{-z})/2}{(e^{z} + e^{-z})/2}$$

$$= \frac{\left(e^{-i(iz)} - e^{i(iz)}\right)/2}{\left(e^{-i(iz)} + e^{i(iz)}\right)/2}$$

$$= \frac{-i\left(e^{i(iz)} - e^{-i(iz)}\right)/2i}{\left(e^{i(iz)} + e^{-i(iz)}\right)/2} = -i\tan(iz).$$

(c) $$\tanh(z) = \frac{1}{i}\left((iz) + \frac{1}{3}(iz)^3 + \frac{2}{15}(iz)^5 + \frac{17}{315}(iz)^7 + \dots\right)$$

$$= z - \frac{1}{3}z^2 + \frac{2}{15}z^5 - \frac{17}{315}z^7 + \dots$$

#2. (a) $$\tan(z) = \frac{\sin z}{\cos z} = \frac{\left(e^{iz} - e^{-iz}\right)/2i}{\left(e^{iz} + e^{-iz}\right)/2} = w$$

$$e^{iz} - e^{-iz} = iw\left(e^{iz} + e^{-iz}\right)$$

$$e^{2iz} - 1 = iw\left(e^{2iz} + 1\right)$$

$$e^{2iz}(1 - iw) = 1 + iw$$

$$e^{2iz} = \frac{1+iw}{1-iw}$$

$$-2iz = \ln\frac{1-iw}{1+iw}$$

$$\arctan(w) = z = \frac{i}{2}\ln\frac{1-iw}{1+iw} = \frac{i}{2}\operatorname{Ln}_p\frac{1-iw}{1+iw} + \pi n$$

arctan(w) has infinitely many values that differ from each other by an arbitrary multiple of π.

(b) : $\arctan\left(\frac{y}{x}\right) = \frac{i}{2} \ln \frac{1-i\frac{y}{x}}{1+i\frac{y}{x}}$

$$= \frac{i}{2} \ln \frac{x-iy}{x+iy} \qquad (z=x+iy)$$

$$= \frac{i}{2}\left(-2i\arg(z) + 2\pi i n\right)$$

$$= \arg(z) + \pi n$$

Thus, $\arg(z) = \arctan\left(\frac{y}{x}\right)$ does not hold in the sense of equality of sets representing values of multivalued fns.

However, it is true that

$$\tan(\arg(z)) = \frac{y}{x}, \text{ when } x = \mathrm{Re}(z) \neq 0$$

and in that sense only

$$\arg(z) = \arctan\left(\frac{y}{x}\right), \text{ for } x \neq 0.$$

#3. The result is trivially true for $s=0$ (the series $1+0+0+\dots$ converges for all $z \in \mathbb{C}$)

For $s \neq 0$ the radius of convergence is 1, since

$\left[a_n = \frac{s(s-1)\dots(s-(n-1))}{n!}\right]$

$$\left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right| = \left|\frac{s-n}{n+1}\right||z| \to |z|, \; n \to \infty$$

For $z \in (-1,1)$, $s \in \mathbb{R}$, $s \neq 0$, the validity of the formula is guaranteed by Taylor's thm.

(since $\left((1+z)^s\right)' = s(1+z)^{s-1}$

$\left((1+z)^s\right)'' = s(s-1)(1+z)^{s-2}$, etc.)

Thus, $$E(s \ln(1+z)) = \sum_{k=0}^{\infty} \frac{s(s-1)\ldots(s-(k-1))}{k!} z^k$$

for $z \in (-1,1)$, $s \in \mathbb{R}$.

For each $z \in \mathbb{C}$, $|z|<1$ the left-hand side is a sum of a convergent series obtained by substitution of

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} + \ldots \quad \text{into}$$

$$E(sw) = 1 + sw + \frac{1}{2}(sw)^2 + \ldots$$

By algebra of power series, the identity holds for all $z \in \mathbb{C}$, $|z|<1$, and for all $s \in \mathbb{C}$!

$$1 + s\left(z - \frac{z^2}{2} + \ldots\right) + \frac{s^2}{2!}\left(z - \frac{z^2}{2} + \ldots\right)^2 + \frac{s^3}{3!}\left(z - \frac{z^2}{2} + \ldots\right)^3 + \ldots$$

$$= 1 + sz + \left(-\frac{s}{2} + \frac{s^2}{2}\right)z^2 + \left(\frac{s}{3} - \frac{s^2}{2} + \frac{s^3}{6}\right)z^3 + \ldots$$

$$= 1 + sz + \frac{s(s-1)}{2!}z^2 + \frac{s(s-1)(s-2)}{3!}z^3 + \ldots$$

Therefore $$(1+z)^s = \sum_{k=0}^{\infty} \frac{s(s-1)\ldots(s-(k-1))}{k!} z^k, \quad |z|<1, \; s \in \mathbb{C}.$$

#4. Circle or straight line:

$$a(x^2+y^2)+bx+cy+d=0$$

$a=0$ - straight line; otherwise

$$a\left(x^2+2\cdot\frac{b}{2a}x+\left(\frac{b}{2a}\right)^2\right)+a\left(y^2+2\cdot\frac{c}{2a}y+\left(\frac{c}{2a}\right)^2\right)$$

$$=\frac{b^2+c^2}{4a}-d>0$$

or $b^2+c^2>4ad$.

Using $x=\frac{z+z^*}{2}$, $y=\frac{z-z^*}{2i}$ obtain

$$a|z|^2+\frac{1}{2}(b-ci)z+\frac{1}{2}(b+ci)z^*+d=0$$

$$\Rightarrow A|z|^2+Bz+B^*z^*+D=0$$

where $A=a$, $D=d$, $B=\frac{1}{2}(b-ci)$

and $|B|^2=\frac{b^2+c^2}{4}>AD$.

If $w=\frac{1}{z}$ then $z=\frac{1}{w}$,

$$A\frac{1}{|w|^2}+B\frac{1}{w}+B^*\frac{1}{w^*}+D=0$$

$$D|w|^2+Bw^*+B^*w+A=0$$

where either $D=0$, or

$$|B^*|^2=|B|^2>AD.$$

(so, straight line or a circle in the w-plane.)

#5. Suppose (x_1, x_2, x_3) and $(-x_1, -x_2, -x_3)$ are two diametrically opposite points on the unit sphere. Then

$$z_1 = \frac{x_1 + i x_2}{1 - x_3}, \quad z_2 = \frac{-x_1 - i x_2}{1 + x_3}$$

and

$$z_1 z_2^* = \frac{(x_1 + i x_2)(-x_1 + i x_2)}{1 - x_3^2} = \frac{-x_1^2 - x_2^2}{1 - x_3^2} = -1.$$

Conversely, suppose $z_2 = \frac{-1}{z_1^*} = \frac{-z_1}{|z_1|^2}$

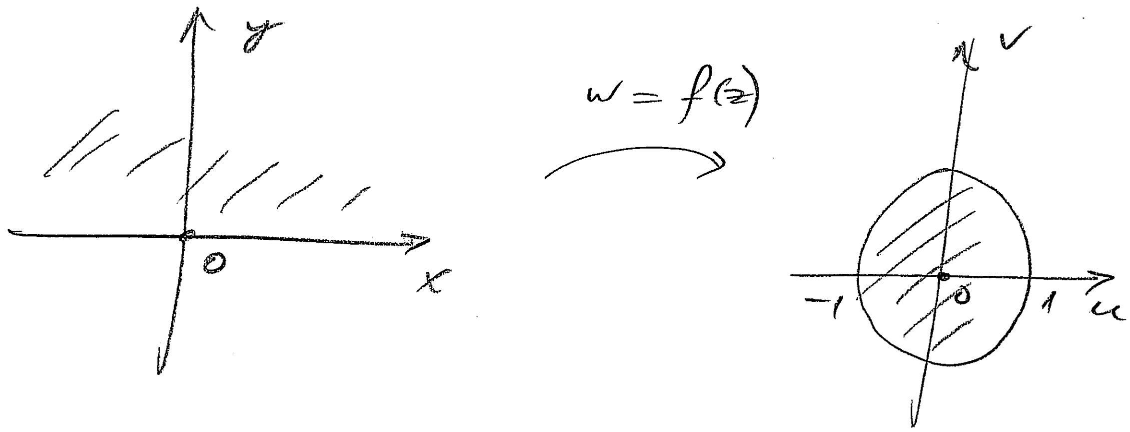
Then
$$x_3^{(2)} = \frac{|z_2|^2 - 1}{|z_2|^2 + 1} = \frac{1 - |z_1|^2}{1 + |z_1|^2} = -x_3$$

$$x_1^{(2)} = \frac{z_2 + z_2^*}{|z_2|^2 + 1} = \frac{-z_1 - z_1^*}{1 + |z_1|^2} = -x_1$$

$$x_2^{(2)} = \frac{z_2 - z_2^*}{|z_2|^2 + 1} = \frac{-z_1 + z_1^*}{1 + |z_1|^2} = -x_2$$

Therefore the stereographic projection of z_2 is the diametrically opposite point to (x_1, x_2, x_3).

#6.



$$0 \longrightarrow -1$$
$$\infty \longrightarrow 1$$
$$ai \longrightarrow 0$$
(a - arbitrary, real.)

(positive imaginary axis should be mapped onto (-1, 1), preserving orthogonality)

$$\frac{w-w_2}{w-w_3} : \frac{w_1-w_2}{w_1-w_3} = \frac{z-z_2}{z-z_3} : \frac{z_1-z_2}{z_1-z_3}$$

$$\frac{w-1}{w-0} : \frac{-1-1}{-1-0} = \frac{z-\infty}{z-ai} : \frac{0-\infty}{0-ai}$$

$$\frac{w-1}{2w} = \frac{-ai}{z-ai}$$

$$\frac{w-1}{w} = \frac{-2ai}{z-ai}$$

$$1-\frac{1}{w} = \frac{-2ai}{z-ai}$$

$$\frac{1}{w} = 1+\frac{2ai}{z-ai} = \frac{z+ai}{z-ai}$$

$$w = \frac{z-ai}{z+ai}$$

Check: if z is real then $\left|\frac{z-ai}{z+ai}\right| = \frac{\sqrt{z^2+a^2}}{\sqrt{z^2+a^2}}$

$= 1$

the real axis is mapped onto the unit circle