February 6, 2019 MATH 655 Prof. V. Panferov

Homework 3

Due on Wed. Feb 13, 2019.

1. (a) Use the known Taylor series expansions for $\sin z$ and $\cos z$ and the method described on page 11 in the textbook to find the first *four* nontrivial terms in the series

$$\tan z = a_0 + a_1 z + \ldots$$

(b) Show that $\tanh(z) = -i\tan(iz)$.

(c) Extend the result of part (a) to the function $\tanh(z)$.

2. (a) By solving the equation $\tan(z) = w$ find an expression for $\arctan(w)$ using the complex logarithm function. Describe the multi-valued character of the function $\arctan(w)$.

(b) Show that

$$\arg(x+iy) = \arctan(y/x), \quad x \neq 0.$$

3. (Problem 1-4:11) Show that the binomial series

$$(1+z)^s = 1 + sz + \frac{s(s-1)}{2!}z^2 + \frac{s(s-1)(s-2)}{3!}z^3 + \ldots$$

converges for $|z| < 1$ and for all $s \in \mathbb{C}$ to the principal value of the left-hand side.

4. Show by direct calculation that the reciprocal function $w = \frac{1}{z}$ transforms circles or straight lines in the complex plane into circles into circles or straight lines. *Hint: show that equation of a straight line or a circle can be written as*

$$A|z|^2 + Bz + B^*z^* + C = 0,$$

where A and C are real and $|B|^2 > AC$. Verify then that the form of the equation is preserved under the transformation $w = \frac{1}{z}$.

5. Show that z_1 and z_2 correspond to diametrically opposite points on the Riemann sphere if and only if $z_1 z_2^* = -1$.

6. (Problem 1-5:1) Use the cross-ratio equality to obtain a mapping which transforms the upper-half z plane into the interior of the unit circle in the w plane.