MATH 655 Homework 2 - Solutions. (1)

#1. Have: $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ abs. conv.

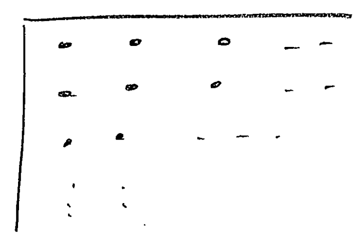
WTS: $\sum_{k=1}^{\infty} (a_1 b_{k-1} + \dots a_{k-1} b_1)$ conv. abs.

and its sum $= \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right)$.

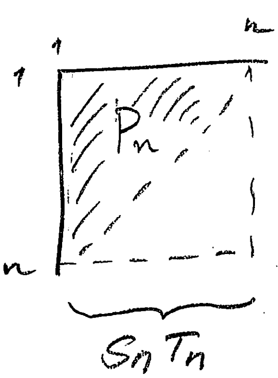
Consider the products:

$$\begin{array}{llll} a_1 b_1 & a_2 b_1 & a_3 b_1 & \dots \\ a_1 b_2 & a_2 b_2 & a_3 b_2 & \dots \\ a_1 b_3 & a_2 b_3 & \dots & \\ \vdots & \vdots & & \end{array}$$

Illustrate by diagram



Let $S_n = \sum_{i=1}^{n} |a_i|$, $T_n = \sum_{i=1}^{n} |b_i|$; $P_n = \sum_{k=1}^{n} (|a_1||b_{k-1}| + \dots + |a_{k-1}||b_1|)$



Then $P_n \leq S_n T_n \leq ST$ $\left(\begin{array}{l} S = \lim S_n \\ T = \lim T_n \end{array}\right)$

$\Rightarrow \sum_{k=1}^{n} |(a_1 b_{k-1} + \dots + a_{k-1} b_1)| \leq P_n$ - bounded

$\Rightarrow$ the series of abs. values $|a_1 b_{k-1} + \dots a_{k-1} b_1|$ converges

Let $s_n = \sum_{i=1}^{n} a_i$, $t_n = \sum_{i=1}^{n} b_i$, $p_n = \sum_{k=1}^{n} (a_1 b_{k-1} + \dots + a_{k-1} b_1)$

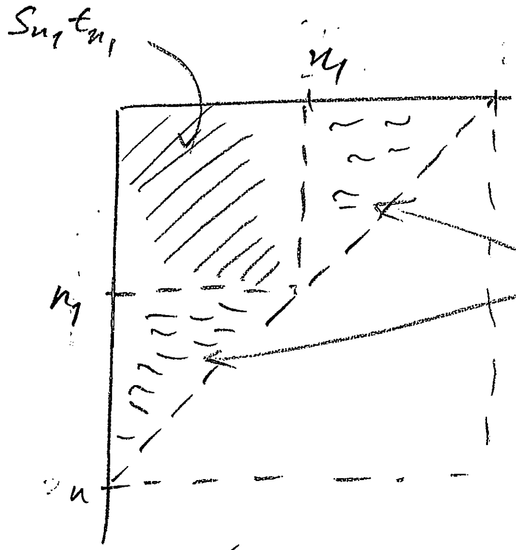
$s = \lim s_n$; $t = \lim t_n$.

Given $\varepsilon > 0$ let $N \in \mathbb{N}$ be such that

$$|s_n t_n - st| < \frac{\varepsilon}{2}$$

and $|S_n T_n - ST| < \frac{\varepsilon}{2}$ for all $n \geq N$.

Claim: $\forall\ n \geq 2N \qquad |p_n - st| < \varepsilon.$



let $n_1 = \left[\frac{n}{2}\right]$ (int. part.)

$$|p_n - st| \leq |S_{n_1} t_{n_1} - st|$$
$$+ |p_n - S_{n_1} t_{n_1}|$$
$$\leq |S_{n_1} t_{n_1} - st| + |S_{n_1} T_{n_1} - ST|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$$\left(|S_{n_1} T_{n_1} - S.T| = \sum_{\substack{i = n_1+1 \\ j = n_1+1}}^{\infty} |a_i||b_j| \right)$$

Therefore $p_n \longrightarrow st$, and

$$\sum_{k=1}^{\infty} (a_1 b_{k-1} + \cdots + a_{k-1} b_1) = \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right).$$

#2. Prove that

$$\lim \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{\infty} \frac{1}{k!}.$$

Soln: Binomial formula:

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!\,n^k}$$
$$= \sum_{k=0}^{n} \frac{1}{k!} \frac{n(n-1)\cdots(n-k+1)}{n^k}$$

$$1 \geq c_{n,k} = \frac{n(n-1)\cdots(n-k+1)}{n^k} \xrightarrow[n\to\infty]{} 1$$

for each k fixed.

Given $\varepsilon > 0$ take $N \in \mathbb{N}$ so large that

$$\left|S_N - e\right| < \frac{\varepsilon}{2}, \quad S_N = 1 + \dots + \frac{1}{N!}, \quad e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

$c_{N,k} \le 1$

Since $c_{N,k} \to 1$ choose $M \in \mathbb{N}, M \ge N$ so large that

$$1 - c_{N,k} < \frac{\varepsilon}{2e}, \qquad k \ge M.$$

Then $\forall\, n \ge M$

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{1}{k!} c_{n,k} \ge \sum_{k=0}^{N} \frac{1}{k!} c_{n,k}$$

$$> \sum_{k=0}^{N} \frac{1}{k!} - \frac{\varepsilon}{2e} \sum_{k=0}^{N} \frac{1}{k!}$$

$$\ge S_N - \frac{\varepsilon}{2}$$

Therefore $\forall\, n \ge M$

$$e - \left(1 + \frac{1}{n}\right)^n \le e - S_N + \frac{\varepsilon}{2} \le \left|e - S_N\right| + \frac{\varepsilon}{2}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\left(1 + \frac{1}{n}\right)^n \le S_n \le e$

we have

$$\left|e - \left(1 + \frac{1}{n}\right)^n\right| < \varepsilon$$

$$\Rightarrow \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e.$$

#3. $\sum_{n=1}^{\infty} a_n z^n$ – series; $R^{-1} = \lim_{n\to\infty} \sup_{m\geq n} |a_m|^{\frac{1}{m}}$

($R = 0, \infty$ if the limit is $\infty, 0$, resp.)

WTS: $\sum_{n=1}^{\infty} a_n z^n$ conv if $|z| < R$
div if $|z| > R$.

case (i). $\lim_{n} \sup_{m\geq n} |a_m|^{\frac{1}{m}} = \rho$ – finite.

Then $\lim_{n} \sup_{m\geq n} |a_m z^m|^{\frac{1}{m}} = |z| \lim_{n} \sup_{m\geq n} |a_m|^{\frac{1}{m}}$

$= |z|\rho \quad < 1$ if $|z| < R$
> 1 if $|z| > R$.

if $|z| < R$, $|z|\rho < 1$ take ε so small
that $|z|\rho + \varepsilon < 1$ (for instance, $\varepsilon = \frac{1-|z|\rho}{2}$)

then $\exists N_\varepsilon \in \mathbb{N} \ \forall n \geq N_\varepsilon$

$\sup_{m\geq n} |a_m z^m|^{\frac{1}{m}} < |z|\rho + \varepsilon$

$\Rightarrow \forall m \geq N_\varepsilon$

$|a_m z^m|^{\frac{1}{m}} < |z|\rho + \varepsilon$

$|a_m z^m| < (|z|\rho + \varepsilon)^m$

$\Rightarrow$ the series $\sum_{N_\varepsilon}^{\infty} |a_n z^n|$ converges
by comparison with convergent geom. series.
$\Rightarrow \sum_{n=1}^{\infty} |a_n z^n|$ converges

If $|z| > R$, $|z|\rho > 1$, take ε so small that $|z|\rho - \varepsilon > 1$.

then $\exists N_\varepsilon \in \mathbb{N}$ $\forall n \geq N_\varepsilon$

$$\sup_{m \geq n} |a_m z^m|^{\frac{1}{m}} > |z|\rho - \varepsilon$$

$$\forall n \geq N_\varepsilon \ \exists m \geq n : |a_m z^m|^{\frac{1}{m}} > |z|\rho - \varepsilon$$

$$\forall n \geq N_\varepsilon \ \exists m \geq n : \quad |a_m z^m| > (|z|\rho - \varepsilon)^m$$

$$\Rightarrow |a_m z^m| \not\to 0$$

$$\Rightarrow \sum_{n=1}^{\infty} a_m z^m \text{ diverges.}$$

case (ii) $\lim_n \sup_{m \geq n} |a_m|^{\frac{1}{m}} = 0$

$$\Rightarrow \lim_n \sup_{m \geq n} |a_m z^m|^{\frac{1}{m}} = 0$$

$$\Rightarrow \forall \varepsilon > 0 \ \exists N_\varepsilon \in \mathbb{N} \ \forall m \geq N_\varepsilon \ |a_m z^m|^{\frac{1}{m}} < \varepsilon$$

$$\Rightarrow |a_m z^m| < \varepsilon^m$$

$\Rightarrow \sum_{n=N_\varepsilon}^{\infty} |a_n z^n|$ converges by comparison with $\sum \varepsilon^n$ for any z.

case (iii) $\lim_n \sup_{m \geq n} |a_m|^{\frac{1}{m}} = \infty$

$$\Rightarrow \lim_n \sup_{m \geq n} |a_m z^m|^{\frac{1}{m}} = \infty$$

$\Rightarrow \exists N_1 \in \mathbb{N} : \forall n \geq N_\varepsilon \ \exists m \geq n$

$|a_m z^m| > 1 \Rightarrow |a_n z^n| \not\to 0$

$\Rightarrow$ the series $\sum a_n z^n$ diverges.

#4 (a) $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} z^n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2(n+1))!\,(n!)^2}{((n+1)!)^2(2n)!} = \frac{(2n+1)(2n+2)}{(n+1)(n+1)} \xrightarrow[n\to\infty]{} 4$$

$\Rightarrow$ radius of convergence $\frac{1}{4}$.

(b) $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!\; n^n}{(n+1)^{n+1}\, n!} = \frac{(n+1)\, n^n}{(n+1)^n (n+1)} = \left(\frac{n}{n+1}\right)^n$$

$$= 1/\left(1+\tfrac{1}{n}\right)^n \longrightarrow 1/e$$

$\Rightarrow$ radius of conv. $= e$.

(c) $\sum_{n=1}^{\infty} 2^{-n} z^{2^n} = \frac{1}{2} z^2 + \frac{1}{4} z^4 + \frac{1}{8} z^8 + \dots$

$$\left(\frac{1}{2^n}\left|z^{2^n}\right|\right)^{\frac{1}{n}} = \frac{1}{2}\,|z|^{\frac{1}{n}2^n} \longrightarrow \begin{cases} 0, & |z|<1 \\ 1, & |z|=1 \\ \infty, & |z|>1 \end{cases}$$

$\Rightarrow$ radius of conv. $= 1$.

(d) $\sum_{n=1}^{\infty} (n+a^n) z^n$

$$\left|\frac{(n+1)+a^{n+1}}{n+a^n}\right| = \left|\frac{1+\frac{1}{n}+\frac{a^{n+1}}{n}}{1+\frac{a^n}{n}}\right| \longrightarrow 1, \quad |a|\le 1$$

$$= |a| \left|\frac{1+\frac{n+1}{a^{n+1}}}{1+\frac{n}{a^n}}\right| \longrightarrow |a|, \quad |a|>1$$

$\Rightarrow$ radius of conv $= 1,\ |a|\le 1;\ 1/|a|,\ |a|>1$.

$a_0 = 1, \ a_1 = \frac{1}{2};$

#5. $\sum_{n=0}^{\infty} a_n z^n$; $a_n = \dfrac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!}$, $n \geq 2$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{2}-n}{n+1}\right| \longrightarrow 1, \ n \to \infty$$

$\Rightarrow$ radius of conv. is $= 1$.

To show $\sum_{n=0}^{\infty} a_n z^n = (1+z)^{\frac{1}{2}}$ use

Taylor expansion of $(1+x)^{\frac{1}{2}}$:

$$(1+x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!} x^n$$

So $$1+x = \left(\sum_{n=0}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!} x^n\right)^2$$

(A direct verification seem difficult!)

Thus, by multiplication formula for power series,

$$a_0 \cdot a_0 = 1$$
$$a_1 a_0 + a_0 a_1 = 1$$
$$a_k a_0 + a_{k-1} a_1 + \cdots + a_0 a_k = 0, \quad k \geq 2.$$

Thus, $1+z = \left(\sum_{n=0}^{\infty} a_n z^n\right)^2$ for all z: $|z|<1$.

#6. $$\frac{2z+3}{z+1} = \frac{2(z+1)+1}{z+1} = 2+\frac{1}{z+1} = 2+\frac{1}{(z-1)+2}$$

$$= 2+\frac{1}{2}\,\frac{1}{1+\left(\frac{z-1}{2}\right)}.$$

Since $$\frac{1}{1-w} = \sum_{k=0}^{\infty} w^k,$$

$$\frac{2z+3}{z+1} = 2+\frac{1}{2}\sum_{k=0}^{\infty}\left(-\left(\frac{z-1}{2}\right)\right)^k$$

$$= 2+\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^k}(z-1)^k.$$

The series converges $\iff \left|\frac{z-1}{2}\right| < 1$

$\iff |z-1| < 2$

The radius of convergence is 2.

#7. (a) $e^{-\frac{\pi}{2}} \in \mathbb{R}$; $e^{\frac{3\pi}{4}i} = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$

$$= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i = \frac{-1+i}{\sqrt{2}}.$$

$$e^{\frac{2\pi}{3}i} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

$$e^{1+i} = e\left(\cos(1) + i\sin(1)\right).$$

(b) $\ln(2) = \ln_p(2) + 2\pi i n$

$\ln(-1) = \pi i + 2\pi i n$

$\ln(i) = \frac{\pi}{2}i + 2\pi i n$

$\ln(-i/2) = -\ln_p(2) - \frac{\pi}{2}i + 2\pi i n$

$\ln(-1-i) = \frac{1}{2}\ln_p(2) - \frac{3\pi}{4}i + 2\pi i n$

$\ln(1+2i) = \frac{1}{2}\ln_p(3) + \arccos\left(\frac{1}{\sqrt{5}}\right)i + 2\pi i n$

#7(c)

$$z^i = e^{i\ln(z)} = e^{i(\ln_p(z) + 2\pi i n)}$$
$$= e^{i\ln_p(z)} e^{-2\pi n}, \quad n \in \mathbb{Z}$$
$$= \left(\cos(\ln_p(z)) + i\sin(\ln_p(z))\right) e^{-2\pi n}, \quad n \in \mathbb{Z}.$$

$$i^i = e^{i\ln(i)} = e^{i\left(\frac{\pi}{2}i + 2\pi i n\right)}$$
$$= e^{-\frac{\pi}{2} - 2\pi n}, \quad n \in \mathbb{Z}$$

$$(-1)^{2i} = e^{2i\ln(-1)} = e^{2i(\pi i + 2\pi i n)}$$
$$= e^{-2\pi - 4\pi n}, \quad n \in \mathbb{Z}.$$

#8.

$$\left(w^{z_1+z_2}\right)_p = e^{(z_1+z_2)\ln_p w}$$
$$\left(w^{z_1}\right)_p \left(w^{z_2}\right)_p = e^{z_1 \ln_p w} e^{z_2 \ln_p w} = e^{(z_1+z_2)\ln_p w} \quad \checkmark$$

$$\left((w_1 w_2)^z\right)_p = e^{z\ln_p(w_1 w_2)}$$
$$= e^{z(\ln_p w_1 + \ln_p w_2 + 2\pi i n)}$$
$$= e^{z\ln_p w_1} e^{z\ln_p w_2} e^{z\cdot 2\pi i n}$$
$$= (w_1)_p^z (w_2)_p^z e^{z\cdot 2\pi i n} \quad \left(\begin{array}{l} n = 0, \pm 1 \\ \text{are possible} \end{array}\right)$$

#9.

$$z^n = e^{n\ln z} = e^{n(\ln_p(z) + 2\pi i k)}$$
$$= e^{n\ln_p(z)} e^{2\pi i k n}$$
$$= e^{n\ln_p(z)}.$$

Thus, z^n is always equal to its principal value.

#10. 1) If we interpret $\ln$ as multivalued logarithm, then it is true that

$$\ln(-1) = \ln\frac{1}{-1} = \ln 1 - \ln(-1) + 2\pi i n$$
$$= -\ln(-1) + 2\pi i n$$

(equality between sets of values.)

Symbolically, we can also write

$$\ln(-1) = -\ln(-1) \qquad \text{(as sets of values.)}$$

The fallacy is then in the word "thus": from the fact that a set is symmetric about 0 it does not follow that the set is zero. In fact

$$\ln(-1) = \pi i + 2\pi i n, \quad n \in \mathbb{Z}$$

satisfies the above relation without being zero.

2) If we try to restrict $\ln$ as single-valued, for instance by considering "$\ln$" $= \ln_p$, then of course the calculation becomes invalid:

$$\ln_p(-1) = \ln_p\frac{1}{-1} \neq \ln_p 1 - \ln_p(-1) = -\ln_p(-1)$$

since $\arg(1)$ and $\arg(1)$ and $\arg(-1)$ differ by π.