February 1, 2019 MATH 655 Prof. V. Panferov

Homework 2

Due on Wed. Feb 6, 2019.

1. Suppose the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent. Show that the series

$$\sum_{k=1}^{\infty} \left(a_1 b_{k-1} + \dots a_{k-1} b_1\right)$$

is absolutely convergent and that its sum equals the product $\left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right)$.

2. Show that

$$\sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Thus, either of the two values can be used as a definition of the number e. *Hint: using the binomial formula,*

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \frac{n(n-1)\dots(n-k+1)}{n^k}.$$

Since

$$1 \geq \frac{n(n-1)\dots(n-k+1)}{n^k} \to 1, \quad n \to \infty$$

for each k fixed, argue that the limit as $n \to \infty$ is the same as the limit of partial sums of the series.

3. Given a series $\sum_{n=1}^{\infty} a_n z^n$ show that its radius of convergence R satisfies

$$R^{-1} = \lim_{n\to\infty} \sup_{m\geq n} |a_m|^{1/m}$$

(if the limit is 0 or ∞ we set R to be ∞ and 0, respectively).

4. Find the radii of convergence for the series:

(a) $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} z^n$

(b) $\sum_{n=1}^{\infty} \frac{(n)!}{n^n} z^n$

(c) $\sum_{n=1}^{\infty} 2^{-n} z^{2^n}$

(d) $\sum_{n=1}^{\infty} (n + a^n) z^n$, $a \in \mathbb{C}$

5. Find the radius of convergence of

$$1+\frac{1}{2}z+\frac{1/2(1/2-1)}{2!}z^2+\frac{1/2(1/2-1)(1/2-2)}{3!}z^3+\dots$$

and prove that the square of this series equals $1+z$.

6. Expand $\dfrac{2z+3}{z+1}$ in powers of $z-1$. What is the radius of convergence?

7. (a) Find the value of e^z for $z=-\frac{\pi}{2}i,\ \frac{3}{4}\pi i,\ \frac{2}{3}\pi i,\ 1+i$.

   (b) Find all values of z such that e^z equal to $2,\ -1,\ i,\ -i/2,\ -1-i,\ 1+2i$.

   (c) Determine all values of $2^i,\ i^i,\ (-1)^{2i}$.

8. Prove that the principal value of $w^{z_1+z_2}$ is equal to the product of the principal values of w^{z_1} and w^{z_2}. What is the relationship between the principal values of w_1^z, w_2^z and $(w_1w_2)^z$?

9. Prove that z^n, n an integer (positive, negative, or zero) is unique.

10. Find the fallacy in the following argument:

$$\ln(-1)=\ln\frac{1}{-1}=\ln 1-\ln(-1)=-\ln(-1)$$

and thus $\ln(-1)=0$.