MATH 655

Homework 1

#1.

$$\frac{2}{1-3i} = \frac{2(1+3i)}{(1-3i)(1+3i)} = \frac{2(1+3i)}{1^2+3^2} = \frac{1}{5} + \frac{3}{5}i$$

$$(1+i\sqrt{3})^6 = \left(2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right)^6 = 2^6\left(\cos 2\pi + i\sin(2\pi)\right)$$
$$= 64$$

$$\frac{1+i}{1-i} = \frac{\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}{\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$$

$$\left(\frac{1+i}{1-i}\right)^5 = i^5 = i$$

$$\frac{1+i\sqrt{3}}{1-i} = \frac{2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}{\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)} = \sqrt{2}\left(\cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\right)$$

$$\left(\frac{1+i\sqrt{3}}{1-i}\right)^4 = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2 + 2\sqrt{3}i.$$

#2.

$$z^* = z^2$$

$$|z^*| = |z| = |z|^2 \Rightarrow |z| = 1 \text{ or } |z| = 0$$

If $z = \cos\theta + i\sin\theta$,

$$z^* = \cos(-\theta) + i\sin(-\theta); \quad z^2 = \cos(2\theta) + i\sin(2\theta)$$

$$\Rightarrow 2\theta = -\theta + 2\pi n$$

$$3\theta = 2\pi n \Rightarrow \theta = \frac{2\pi}{3}n, \quad n \in \mathbb{Z}$$

Unique values: $n = 0 \Rightarrow z = 1$

$n = 1 \Rightarrow z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$n = 2 \Rightarrow z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

also $z = 0$.

#3. $n=2$:

$|z_1+z_2| \le |z_1|+|z_2|$ proved in class.

Suppose for $k \in \mathbb{N}$

$|z_1+\dots+z_k| \le |z_1|+\dots+|z_k|$

Then

$|z_1+\dots+z_k+z_{k+1}| \le |z_1+\dots+z_{k}|+|z_{k+1}|$

$\le |z_1|+\dots+|z_{k}|+|z_{k+1}|$

Equality: $|z_1+z_2|^2 = |z_1|^2+|z_2|^2+2\,Re\,(z_1^* z_2)$

$= (|z_1|+|z_2|)^2 = |z_1|^2+|z_2|^2+2|z_1||z_2|$

$\iff |z_1||z_2| = Re\,(z_1^* z_2)$

$\iff |z_1^* z_2| = Re\,(z_1^* z_2)$

$\iff z_1=0$ or $z_2=0$ or $z_1^* z_2 > 0$ (real)

$\iff arg(z_1^*) = -arg(z_2)$

$\iff arg(z_1) = arg(z_2)$

$\iff z_2 = c z_1$ or $z_1 = c z_2$, $c \ge 0$.

Suppose $|z_1+\dots+z_n| = |z_1|+\dots+|z_n|$

WLOG $\forall i\ z_i \ne 0$; otherwise the value n is reduced.

WTS $\forall i,j\ arg(z_i) = arg(z_j)$

Induction: $n=2$ done above.

Suppose true for $n=k$

$|z_1+\dots+z_{k+1}| \le |z_1+\dots+z_k|+|z_{k+1}|$

$\le |z_1|+\dots+|z_k|+|z_{k+1}|$

$\Rightarrow |z_1+\dots+z_k| = |z_1|+\dots+|z_k| \Rightarrow arg(z_1) = \dots = arg(z_k)$

and $arg(z_{k+1}) = arg(z_1+\dots+z_k)$

$= arg(z_1) = \dots = arg(z_k)$.

#4. $\xi^n = 1$; $\xi = |\xi|(\cos\theta + i\sin\theta)$

$|\xi^n| = |\xi|^n = 1 \Rightarrow |\xi| = 1$;

$$\xi^n = |\xi|^n (\cos(n\theta) + i\sin(n\theta)) = 1 + i0$$

$$\Rightarrow \cos(n\theta) = 1, \ \sin(n\theta) = 0$$

$$\theta = 0, \frac{2\pi}{n}, \dots \frac{2\pi(n-1)}{n} + 2\pi m, \ m \in \mathbb{Z}$$

n solutions $\xi = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$

$k = 0, 1 .. n-1$

— n distinct solutions.

#5. (a) Real form: $y - y_1 = m(x - x_1)$

$$(y - y_1)(x_2 - x_1) = (y_2 - y_1)(x - x_1)$$

Cplx form: $(x+iy)(x_1 - x_2 - i(y_1 - y_2)) + (x_1 + iy_1)(x_2 - x - i(y_2 - y))$
$+ (x_2 + iy_2)(x - x_1 - i(y - y_1)) = 0$

Real part: $x(x_1 - x_2) + y(y - y_2)$

$x_1(x_2 - x) + y_1(y_2 - y)$

$x_2(x - x_1) + y_2(y - y_1) = 0$ (cancellation)

Im. part:

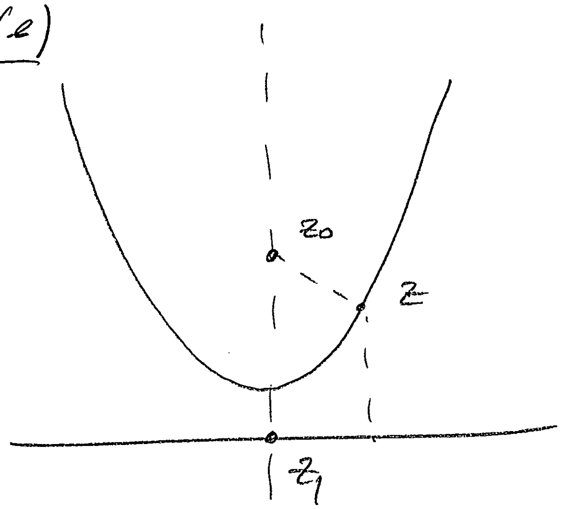
$$0 = \left(\begin{array}{l} y(x_1 - x_2) - x(y_1 - y_2) \\ + y_1(x_2 - x) - x_1(y_2 - y) \\ + y_2(x - x_1) - x_2(y - y_1) \end{array} \right\} = 0$$

From real form: $y_2(x - x_1) + y(x_1 - x_2) = y_1(x - x_1) + y_1(x_1 - x_2)$

$= y_1(x - x_2)$

$(y_2 - y_1)x - x_2(y - y_1) = (y_1 - y)x_1 + (y_2 - y_1)x_1 = x_1(y_2 - y)$

#5(e)



(4)

$|z - z_0| = d$ (distance from a point to a line)

Directrix:

$$(x-x_1)(x_0-x_1)+(y-y_1)(y_0-y_1)=0$$

$$d = \frac{|(x-x_1)(x_0-x_1)+(y-y_1)(y_0-y_1)|}{|z_0-z_1|}$$

$$= \frac{|\mathrm{Re}\,(z-z_1)(z_0-z_1)^*|}{|z_0-z_1|}$$

$$|z-z_0|\,|z_0-z_1| = |\mathrm{Re}\,(z-z_1)(z_0-z_1)^*|$$

$$= \frac{1}{2}\left((z-z_1)(z_0-z_1)^* + (z-z_1)^*(z_0-z_1)\right)$$

#6

$$1+z+\dots+z^n = \frac{1-z^{n+1}}{1-z}, \quad z = r(\cos\theta + i\sin\theta)$$

$$\mathrm{Re}\,(1+z+\dots+z^n) = 1 + r\cos\theta + r^2\cos(2\theta) + \dots$$
$$\dots + r^n\cos(n\theta)$$

Explicitly:

$$\mathrm{Re}\left(\frac{1-z^{n+1}}{1-z}\right) = \mathrm{Re}\,\frac{1-r^{n+1}(\cos((n+1)\theta)+i\sin((n+1)\theta))}{(1-r\cos\theta)-ir\sin\theta}$$

$$= \mathrm{Re}\,\frac{(1-r^{n+1}\cos((n+1)\theta)-ir^{n+1}\sin((n+1)\theta))((1-r\cos\theta)+ir\sin\theta)}{(1-r\cos\theta)^2+r^2\sin^2\theta}$$

$$= \frac{(1-r^{n+1}\cos((n+1)\theta))(1-r\cos\theta)+r^{n+2}\sin((n+1)\theta)\sin\theta}{1-2r\cos\theta+r^2}$$

$$= \frac{1-r\cos\theta-r^{n+1}\cos((n+1)\theta)+r^{n+2}\cos((n+2)\theta)}{1-2r\cos\theta+r^2}$$

#7(a) $\sum_{n=1}^{\infty} z_n$ conv. abs $\Rightarrow \sum_{n=1}^{\infty} |z_n|$ conv.

(5)

$z_n = u_n + i v_n$; $|z_n| = \sqrt{u_n^2 + v_n^2}$

$|u_n| \leq |z_n|$; $|v_n| \leq |z_n|$

By comparison test $\sum_{n=1}^{\infty} u_n , \sum_{n=1}^{\infty} v_n$ conv.

#7(b) $s: \mathbb{N} \to \mathbb{N}$ – bijection.

Suppose $\sum |z_n|$ conv.

WTS : $\sum |z_{s(n)}|$ conv. Let $S_n = \sum_{i=1}^{n} |z_i|$.

Suffices to show $S_n' = \sum_{i=1}^{n} |z_{s(i)}|$ bounded above.

But $S_n' \leq S_m \leq \sum_{n=1}^{\infty} |z_n|$, where $m = \max(s\{1 \ldots n\})$.

Further, WTS $\sum_{i=1}^{\infty} z_i = \sum_{i=1}^{\infty} z_{s(i)}$.

Let $M = \sum_{i=1}^{\infty} z_i$, $s_n = \sum_{i=1}^{n} z_i$, $L = \sum_{i=1}^{\infty} |z_i|$

Given $\varepsilon > 0$ take $N_\varepsilon \in \mathbb{N}$ $\forall n \geq N_\varepsilon$

$|s_n - M| < \frac{\varepsilon}{2}$, $|S_n - L| < \frac{\varepsilon}{2}$

Take $K = \max(s^{-1}\{1, \ldots N\})$

Then $\forall k \geq K$

$$s_k' = \sum_{i=1}^{k} z_{s(i)} = \sum_{i=1}^{N} z_i + \sum_{\substack{i=1 \\ \sigma(i) > N}}^{k} z_{s(i)}$$

$$|s_k' - M| \leq |s_N - M| + \sum_{\substack{i=1 \\ s(i) > N}}^{k} |z_{s(i)}|$$

$$\leq |s_N - M| + \sum_{i=N+1}^{\infty} |z_i|$$

$$\leq |s_N - M| + |S_N - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon.$$

#8. $f_\varepsilon(x) = \dfrac{\varepsilon}{\varepsilon^2+x^2} \longrightarrow \begin{cases} 0, & x\neq 0 \\ \infty, & x=0 \end{cases}$

$$f(x) = 0, \quad x\neq 0.$$

on $\mathbb{R}\setminus\{0\}$ the conv. is not uniform,

since $f_\varepsilon(\varepsilon) = \dfrac{\varepsilon}{2\varepsilon^2} = \dfrac{1}{2\varepsilon} \longrightarrow \infty, \quad \varepsilon\to 0.$

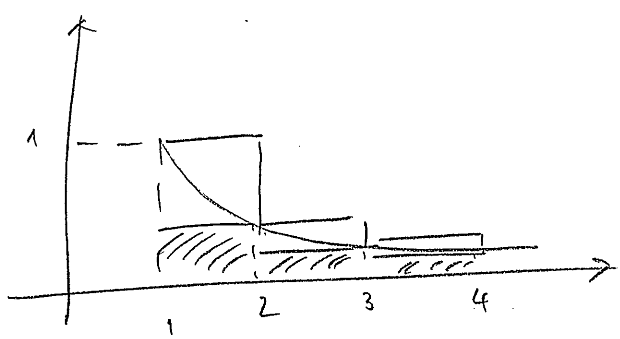
on $(-\infty,-c_0] \cup [c_0,\infty)$ $(c_0>0)$

the convergence is uniform:

$$|f_\varepsilon(x)-f(x)| = \frac{\varepsilon}{\varepsilon^2+x^2} \le \frac{\varepsilon}{\varepsilon^2+c_0^2} \le \frac{\varepsilon}{c_0^2} < \eta$$

if $\varepsilon < c_0^2\eta$.

#9.

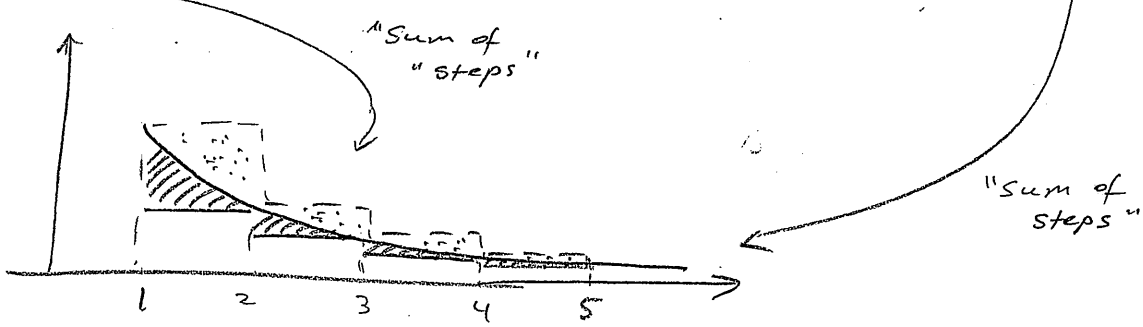


$$1+\frac{1}{2}+\dots+\frac{1}{n} > \int_1^{n+1}\frac{1}{x}\,dx = \ln(n+1)$$

$$\frac{1}{2}+\dots+\frac{1}{n} < \int_1^{n}\frac{1}{x}\,dx = \ln(n)$$

$$0 < \underbrace{1+\dots+\frac{1}{n} - \ln(n+1)}_{\text{bounded above, nondecreasing}} < \underbrace{1+\frac{1}{2}+\dots+\frac{1}{n}}_{\text{bounded above, nonincreasing}} - \ln n < 1$$

since $1+\frac{1}{2}+\dots+\frac{1}{n} - \ln n = 1 - \Big(\underbrace{\ln n - \frac{1}{2} - \frac{1}{3} - \dots - \frac{1}{n}}_{\text{nondecreasing } \ge 0}\Big)$



Therefore $1+\frac{1}{2}+\dots+\frac{1}{n} - \ln n \longrightarrow \gamma$

where $0<\gamma<1.$