Homework 1

Due on Wed. Jan 30, 2019.

1. Find the real and imaginary parts of:

$$\frac{2}{1-3i}, \quad (1+i\sqrt{3})^6, \quad \left(\frac{1+i}{1-i}\right)^5, \quad \left(\frac{1+i\sqrt{3}}{1-i}\right)^4.$$

2. Find all complex z satisfying $z^* = z^2$.

3. Use induction to prove the general version of the triangle inequality:

$$|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|.$$

   Describe completely the conditions under which equality is achieved.

4. If n is a positive integer, show that there are exactly n distinct roots of unity (i. e. solutions of the equation $\zeta^n = 1$) and that geometrically these roots form the vertices of a regular n-gon.

5. (a) Show that the equation of a straight line through z_1 and z_2 is

$$z(z_1 - z_2)^* + z_1(z_2 - z)^* + z_2(z - z_1)^* = 0.$$

   (b)* A parabola is defined geometrically as a locus of points in a plane which are an equal distance away from a given point (focus) and given line (directrix). Derive a complex form similar to one in part (a) for a parabola with focus at z_0 and directrix passing through z_1, perpendicular to $z_1 - z_0$.

6. Show that if r is a real number,

$$1 + r\cos\theta + r^2\cos 2\theta + \cdots + r^n \cos n\theta$$

   can be found as the real part of $(1 - z^{n+1})/(1 - z)$, where

$$z = r(\cos\theta + i\sin\theta).$$

   Obtain this real part explicitly.

7. (a) Suppose $\sum_{i=1}^{\infty} z_i$ converges absolutely. Let $u_i = \mathrm{Re}(z_i)$, $v_i = \mathrm{Im}(z_i)$. Show that $\sum_{i=1}^{\infty} u_i$, $\sum_{i=1}^{\infty} v_i$ are absolutely convergent.

(b)* A rearrangement of indices is a bijection $s : \mathbb{N} \to \mathbb{N}$. If $\sum_{i=1}^{\infty} z_i$ is absolutely convergent, show that a rearranged series $\sum_{i=1}^{\infty} z_{s(i)}$ is absolutely convergent, and the corresponding sums are equal. Give an example to show that the same property is no longer true if a series converges conditionally.

8. Find the limit function of $\varepsilon/(\varepsilon^2 + x^2)$ as $\varepsilon \to 0$, where x and ε are real. For what region is the convergence uniform?

9. Use an argument of a Cauchy-integral-test kind to show that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n$$

approaches a constant (Euler's constant γ) as $n \to \infty$.