#1. Suppose $f: D' \to D$ is not conformal at $\xi_0 \in D'$:

$$f'(\xi_0) = 0, \quad \text{and } z_0 = f(\xi_0).$$

Let
$$f(\xi) = f(\xi_0) + \frac{1}{m!} f^{(m)}(\xi_0)(\xi - \xi_0)^m + \dots,$$
so m is the order of first non-vanishing derivative at ξ_0.

By Identity Theorem, ξ_0 cannot be a point of accumulation of zeros of $f(\xi) - z_0$, as well as of $f'(\xi)$.

Therefore, for $\varepsilon > 0$ small enough

$$f(\xi) \neq z_0; \quad f'(\xi) \neq 0 \quad \text{for } \xi: \ 0 < |\xi - \xi_0| \leq \varepsilon.$$

The function $h(\xi) = |f(\xi) - z_0|$ achieves a positive minimum on $\xi: |\xi - \xi_0| = \varepsilon$; denote this positive value δ.

Now compare the two equations:

$$f(\xi) - z_0 = 0$$

and $f(\xi) - z = 0$, for z close enough to z_0.

The first one has m solutions counting with multiplicity (actually, one solution of mult. m.)

Let $z: 0<|z-z_0|<\delta$.

Then $\left|(f(\xi)-z_0)-(f(\xi)-z)\right| = |z-z_0| < \delta \le |f(\xi)-z_0|$

for $\xi: |\xi-\xi_0| = \varepsilon$.

By Rouche's Theorem the function $f(\xi)-z$ must have the same # of zeros counting with multiplicity as the first eqn. (i.e. m.)

However, ξ_0 is not a solution of $f(\xi)-z=0$ (since $z \neq z_0$)

and $f'(\xi) \neq 0$ for $\xi: 0<|\xi-\xi_0| \le \varepsilon$

$\Rightarrow$ every solution of $f(\xi)-z$ is simple.

Thus, $f(\xi)=z$

has exactly m distinct solutions

in $|\xi-\xi_0|<\varepsilon$ given that $0<|z-z_0|<\delta$.

$\Rightarrow f$ is not one-to-one.

#2. Denote by D the interior of C.

(Jordan's Theorem: C is the topological bdry of D.)

Let $F: D \cup C \to \mathbb{C}$ be analytic.

The following facts are true:

* $F(D)$ is open in $\mathbb{C}$ (The Open Mapping Theorem.)

* $F(D)$ is a region in $\mathbb{C}$ (continuous image of a connected set is connected.)

* $F(C)$ is a contour in $\mathbb{C}$ (by composition, $F(C)$ is a continuous image of an interval; $F(C)$ is piecewise smooth if C is piecewise smooth.)

* $F(C)$ is the boundary of $F(D)$.

  (F is continuous $\Rightarrow$ the image of a bdry pt in D is a bdry pt in $F(D)$.
  Also, by the Open Mapping Theorem, the inverse image of any bdry pt in $F(D)$ must come from a bdry pt of D, since D has no isolated pts, and interior pts are mapped into interior pts.)

Since F assumes any value at most once on C, $F(C)$ is a simple contour (without self-intersections.)

Therefore (Jordan's Theorem) $\mathbb{C} = F(C) \cup I \cup E$ where I and E are two regions having $F(C)$ as their boundary, I is bounded and E is unbounded.

$F(D)$ and E are regions, so if $F(D) \subseteq E$ then $F(C) = \text{bdry}(F(D)) = \text{bdry}(E)$ implies $F(D) = E$. Indeed, if $E \setminus F(D) \neq \emptyset$, connect a point in that set with a point in $F(D)$ by a path $\gamma(t)$ and define

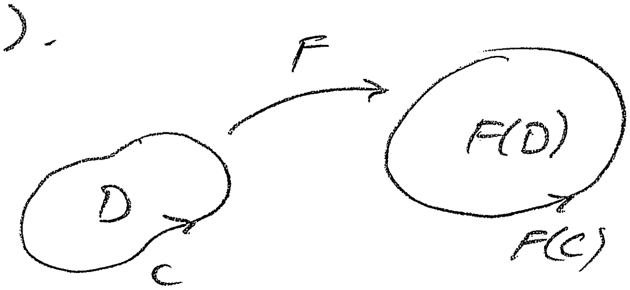
$$\varphi(t) = \begin{cases} 0, & \gamma(t) \notin F(D) \\ 1, & \gamma(t) \in F(D) \end{cases}$$

Then φ must be discontinuous for at least one value $t = t_0$, so $\gamma(t_0) \in \text{bdry}(F(D))$, which is impossible since $\gamma(t_0)$ is interior to E, and $\text{bdry}(E) = \text{bdry}(F(D)) \Rightarrow \gamma(t_0) \notin \text{bdry}(F(D))$.

However $F(D) = E$ is impossible since $F(D)$ is bounded, while E is not.

Therefore $F(D) \subseteq I$ and then $F(D) = I$ by the previous argument.

Thus $F(D)$ is the region exterior to the simple contour $F(C)$.



Since F is conformal, it preserves the orientation of C:
Rotating tangent vector to C by $90°$ leaves us within D, the same must be true for $F(D)$.

If $z_0 \in F(D)$ then $(\xi) - z_0 \neq 0$ on C,
so

$$\#\,\text{zeros}\left(F(\xi) - z_0\right) = \int_C \frac{F'(\xi)\,d\xi}{F(\xi) - z_0}$$

$$= \int_{F(C)} \frac{dz}{z - z_0} = W\left(F(C), z_0\right)$$

$$= 1$$

since the winding number is 1 for a simple contour oriented positively.

Thus $F(\xi)$ assumes any value at most once outside C $\Rightarrow$ F is one-to-one.

$\Rightarrow$ F is a bijection of D' onto D.

By Problem 1, F is also conformal on D'.

#3. $\cos z = \sinh \xi$.

$\arccos w$ has branch points at $w = \pm 1$;

$\sinh \xi = 1$

$e^{\xi} - e^{-\xi} = 2$

$e^{2\xi} - 2e^{\xi} - 1 = 0$

$(e^{\xi} - 1)^2 - 2 = 0$

$e^{\xi} = 1 \pm \sqrt{2}$

$\xi = \ln(1 \pm \sqrt{2})$

$= \begin{cases} \ln_p(1+\sqrt{2}) + 2\pi n i \\ \ln_p \frac{1}{\sqrt{2}-1} + i\pi + 2\pi n i \end{cases}$

$\sinh \xi = -1$:

$e^{\xi} - e^{-\xi} = -2$

$e^{2\xi} + 2e^{\xi} - 1 = 0$

$(e^{\xi} + 1)^2 - 2 = 0$

$e^{\xi} = -1 \pm \sqrt{2}$

$\xi = \ln(-1 \pm \sqrt{2})$

$= \begin{cases} \ln_p(\sqrt{2}-1) + 2\pi i n \\ \ln_p \frac{1}{1+\sqrt{2}} + i\pi + 2\pi i n \end{cases}$

$(\sinh \xi)' = \cosh \xi \neq 0$ when $\sinh \xi = \pm 1$

(since $\cosh^2 \xi = \sinh^2 \xi + 1 = 2$)

Therefore $\sinh \xi$ is conformal in a neighborhood of any ξ: $\sinh \xi = \pm 1$

$\Rightarrow$ image of any single closed contour about each of these values is a single contour about $w = \pm 1$.

Therefore $z = \arccos(\sinh \xi)$ has a branch point (i.e. non-analytic) at each ξ: $\sinh \xi = \pm 1$.

$\Rightarrow$ all these values are critical points.

where is $\frac{dz}{d\xi} = 0$?

Assuming $\frac{dz}{d\xi}$ exists,

$$(\sin z)\,\frac{dz}{d\xi} = \cosh\xi$$

$$\Rightarrow \frac{dz}{d\xi} = \frac{\cosh\xi}{\sin z}\,;$$

$\sin z \neq 0$ if $\sinh\xi \neq \pm 1$.

$\cosh\xi = 0$ if

$$\xi = \frac{\pi i}{2} + i\pi n\,,\; n \in \mathbb{Z}$$

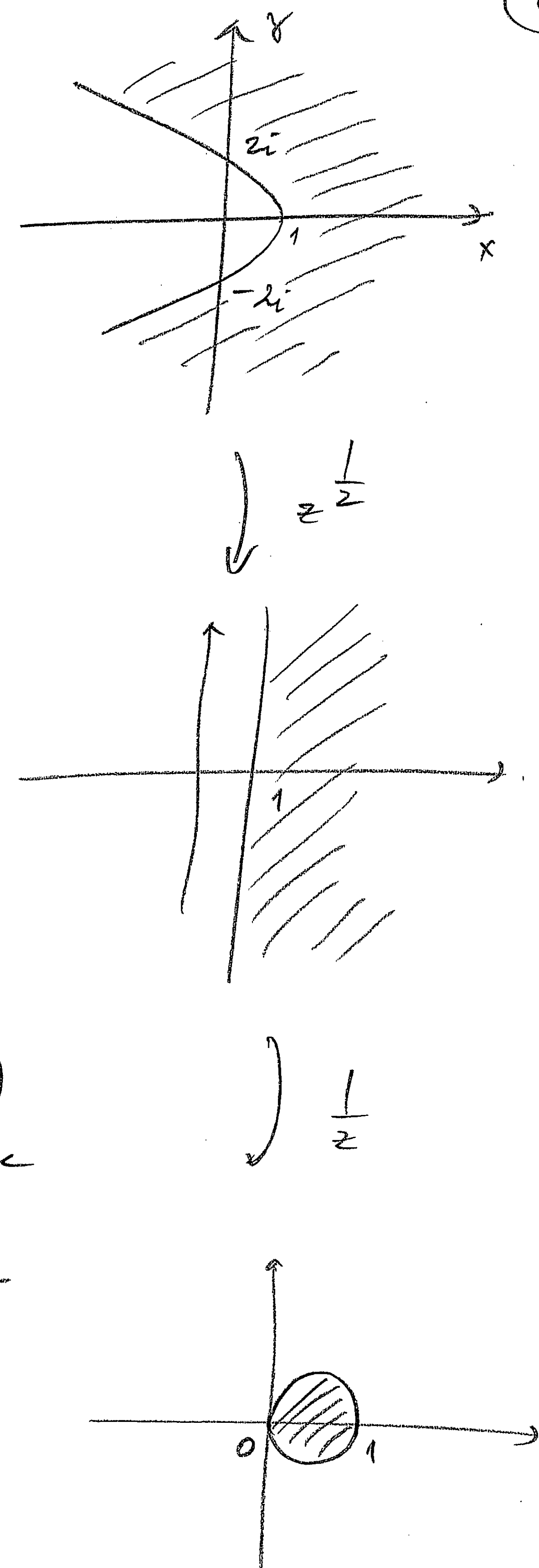
Thus the mapping has critical points

at $\xi = \ln\left(\pm\left(1 \pm \sqrt{2}\right)\right)$

and $\xi = \pi i\left(n + \frac{1}{2}\right)$, $n \in \mathbb{Z}$.

#4 (a)

$$y^2 = 4 - 4x$$
$$4x = 4 - y^2$$
$$x = 1 - \left(\tfrac{y}{2}\right)^2$$



If $\operatorname{Re}\xi = 1$ then

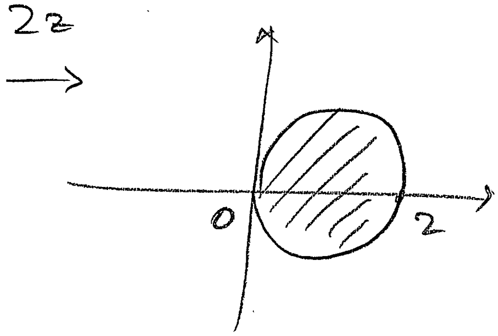
$$\xi^2 = (1+i\eta)^2 = 1 - \eta^2 + 2i\eta$$
$$= x + iy$$
$$\Rightarrow \; y^2 = 4\eta^2 = 4 - 4(1-\eta^2)$$
$$= 4 - 4x^2$$

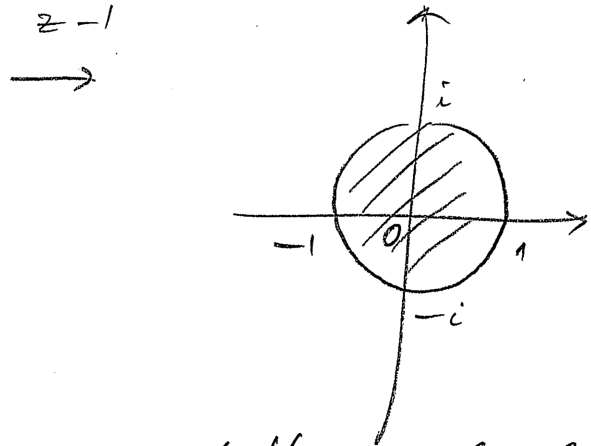
So the image of $\{\operatorname{Re}\xi = 1\}$ under the ξ^2 map is the parabola $y^2 = 4 - 4x$

$\Rightarrow$ for a branch of $z^{\frac{1}{2}}$ with a cut (for example) along $(-\infty, 0]$ the image of the region exterior to the parabola is the half-plane $\{\operatorname{Re}\xi > 1\}$.

The image of $\{\operatorname{Re}\xi = 1\}$ is the circle through 0 and 1, symmetric about the x-axis:

$$|w - \tfrac{1}{2}| = 1$$

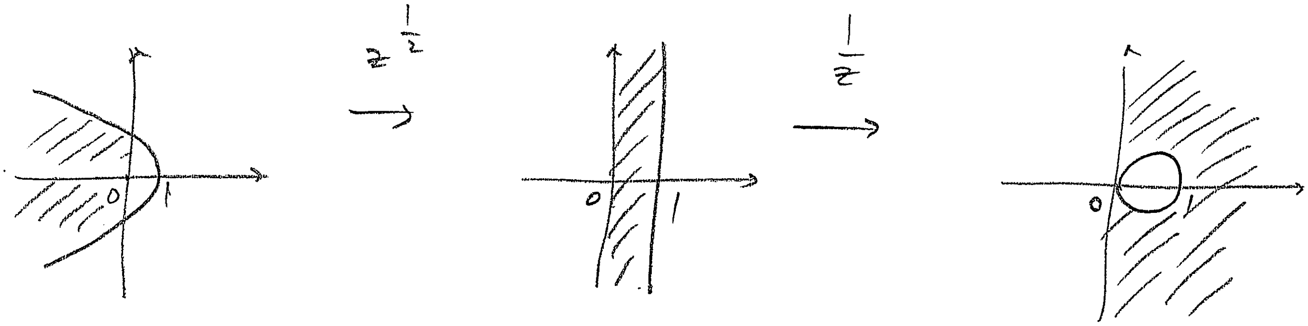




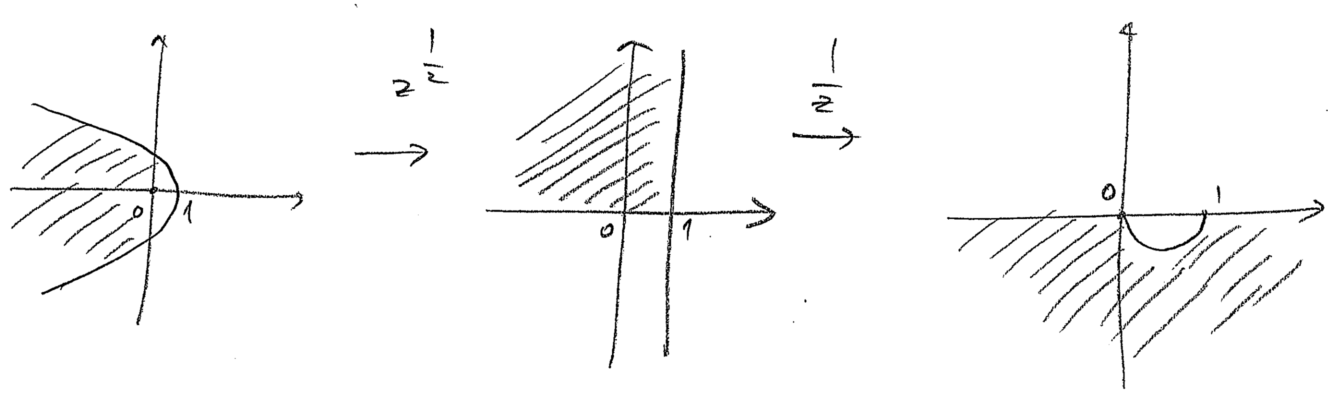
Scale by a factor of 2

shift to the left by 1.

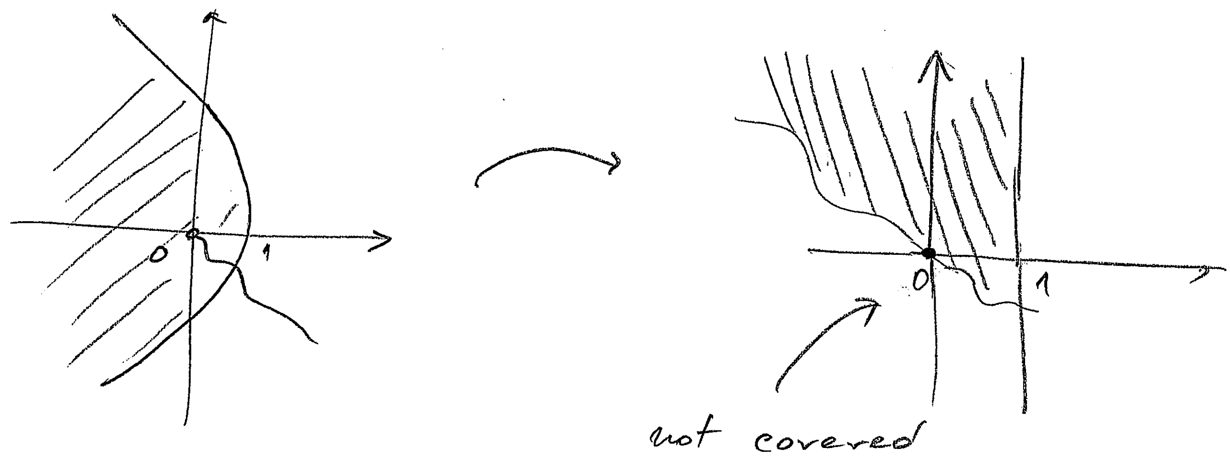
By using the same branch of $z^{\frac{1}{2}}$ the region exterior to the parabola is mapped onto $\{\xi : 0 < \operatorname{Re} \xi < 1\}$:



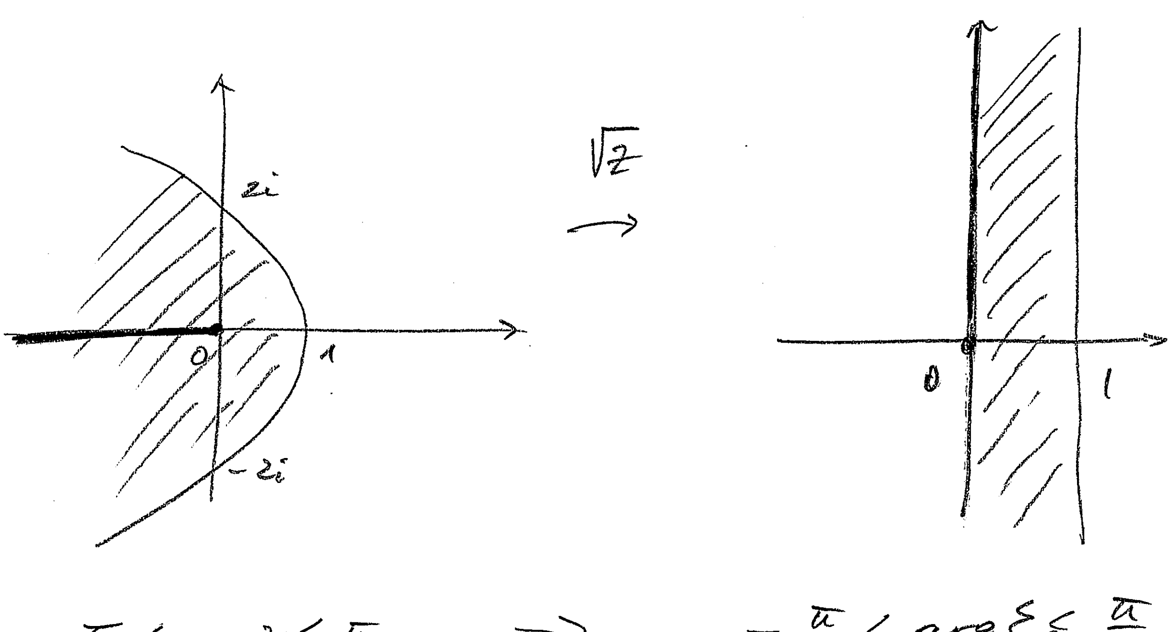
using the branch with a cut along $[0,\infty)$ we get



In general, any branch of $z^{\frac{1}{2}}$ will have a cut through $z=0$ (exterior to the parabola) so near $z=0$ the image will look like half-plane $\Rightarrow$ impossible to cover $\{\xi : \mathrm{Re}\,\xi < 1\}$ as an image of the exterior of the parabola with any branch of $z^{\frac{1}{2}}$

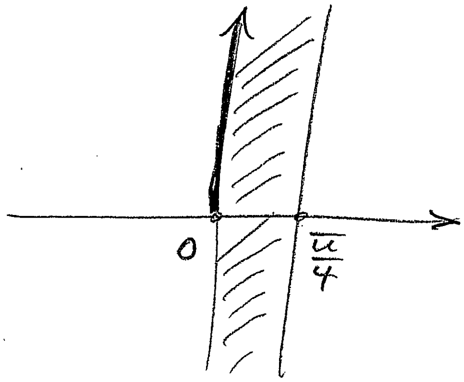


(b) $\xi = \tan^2 \frac{\pi\sqrt{z}}{4}$



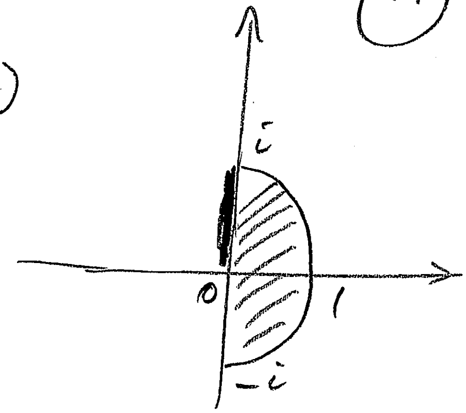
$-\pi < \arg z \le \pi \quad \Rightarrow \quad -\frac{\pi}{2} < \arg \xi \le \frac{\pi}{2}$

$\xrightarrow{\frac{\pi}{4}z}$

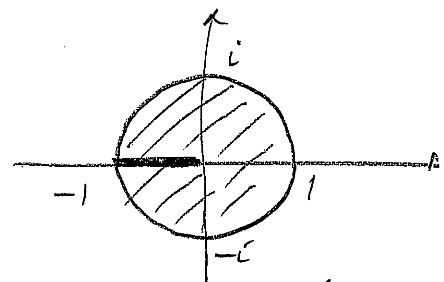


Scale by $\frac{\pi}{4}$

$\xrightarrow{\tan z}$ (*)



z^2

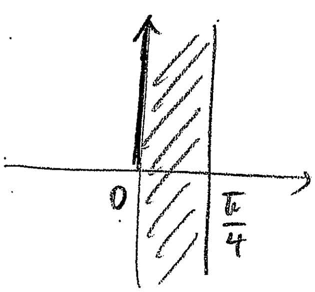


(bingo!)

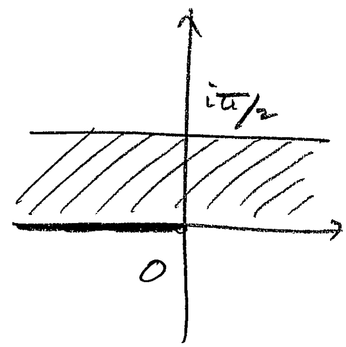
(*) Indeed, $\tan z = \dfrac{2}{2i}\,\dfrac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}$

$$= -i\,\frac{e^{2iz}-1}{e^{2iz}+1}$$

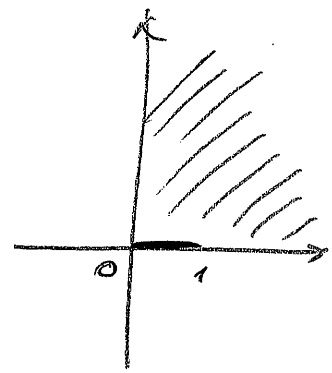
So



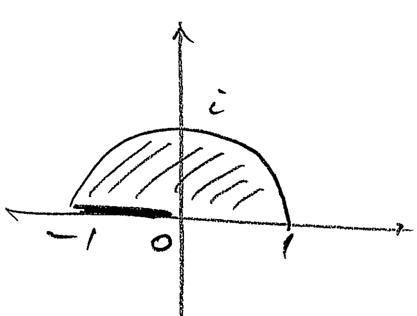
$\xrightarrow{2iz}$



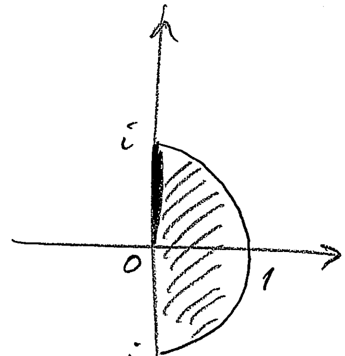
$\xrightarrow{e^z}$



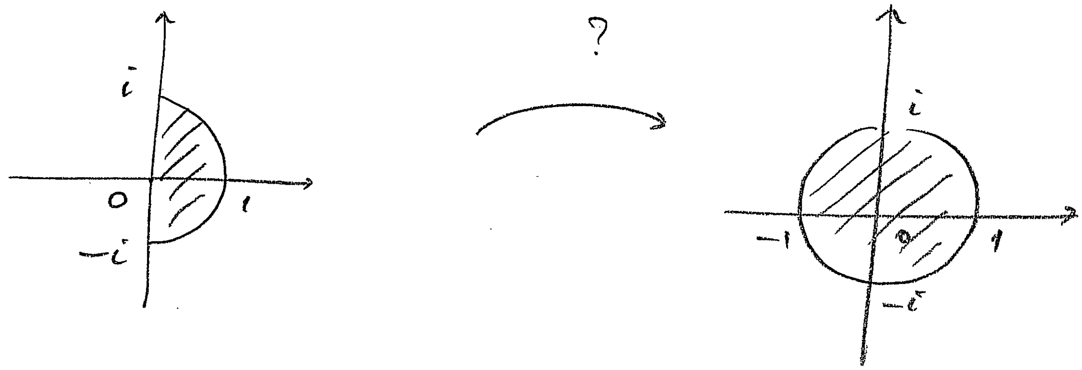
$\xrightarrow{\frac{z-1}{z+1}}$



$\xrightarrow{-iz}$



#5. Semidisk $|z|<1$, $\mathrm{Re}(z)>0$ onto $|z|<1$.



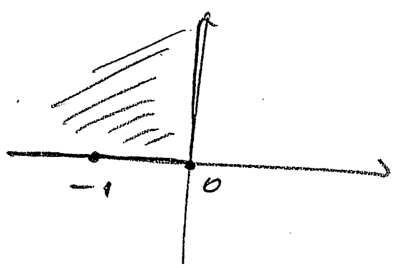
$\Big\downarrow \ \dfrac{z+i}{z-i}$

$(0, -i, i) \longrightarrow (-1, 0, \infty)$

line segment from $-i$ to i maps onto $(-\infty, 0]$

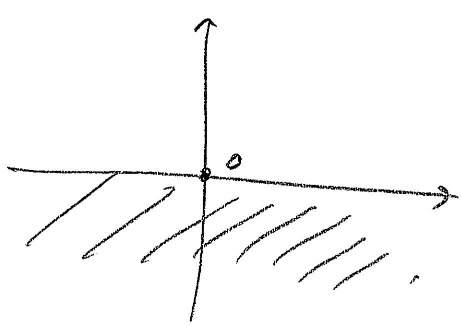
angles preserved $\Rightarrow$ half-circle $|z|=1$, $\mathrm{Re}(z)>0$ maps into $(0, i\infty)$.

interior mapped onto 2^{nd} quadrant.



$\Big\downarrow \ z^2$

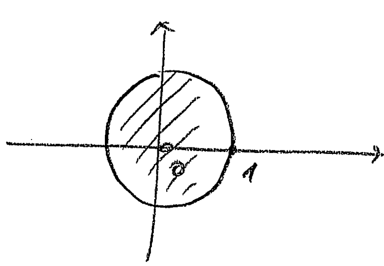
doubles the argument for every z.



$\Big\downarrow \ \dfrac{z+i}{z-i}$

$(-i, i, \infty) \rightarrow (0, \infty, 1)$

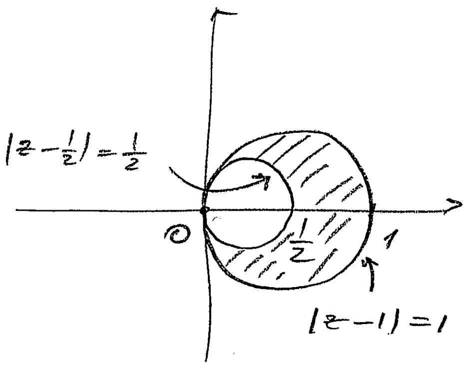
real axis $\rightarrow$ unit circle



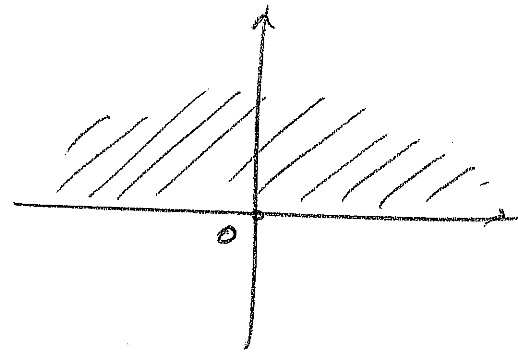
Composition:

$$w = \frac{\left(\frac{z+i}{z-i}\right)^2 + i}{\left(\frac{z+i}{z-i}\right)^2 - i}$$

#6.



$\xrightarrow{?}$

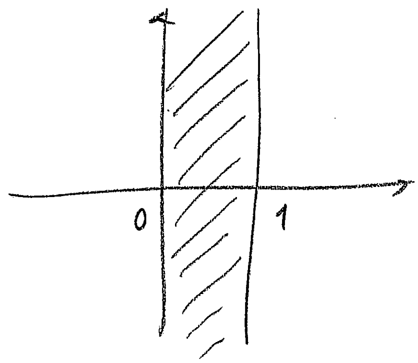


$$\Big\downarrow \frac{1-z}{z}$$

maps $1 \longrightarrow 0$, $0 \longrightarrow \infty$

$\frac{1}{2} \longrightarrow 1$

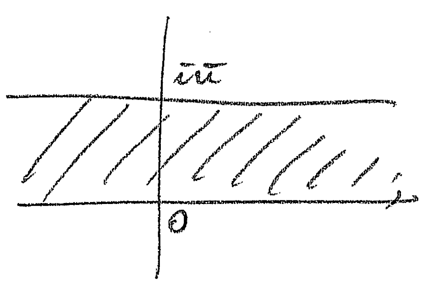
preserves symmetry about the real axis.



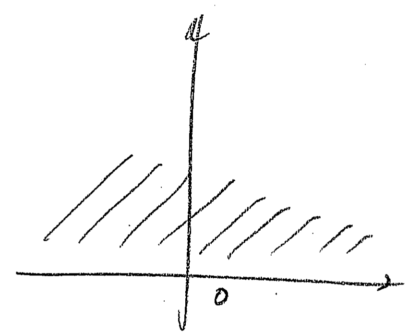
$$\Big\downarrow i\pi z$$

Scales by a factor π

rotates positively by 90°.



$\xrightarrow{}$ e^z



$$e^{x+iy} = e^x(\cos y + i \sin y)$$

$\Rightarrow$ distance from 0 arbitrary; argument in $(0,\pi)$.

Composition:

$$w = e^{i\pi\frac{1-z}{z}}$$

#7. We assume that F extends to the boundary of $A_1 = \{z : 1 < |z| < R_1\}$ in such a way that $\lim\limits_{|z|\to 1} F(z) = 1$ and $\lim\limits_{|z|\to R_1} F(z) = R_2$ (if the order of $1, R_2$ is reversed, we can consider $\tilde{F} = R_2/F$).

[See Ahlfors p. 232]

Consider $p(z) = \ln R_1 \ln F(z) - \ln R_2 \ln z$.

then $Re\,(p(z)) = \ln R_1 \ln |F(z)| - \ln R_2 \ln |z|$ is harmonic on A_1 and satisfies the boundary conditions

$$Re\,(p(z))\Big|_{|z|=1} = 0, \quad Re\,(p(z))\Big|_{|z|=R_1} = 0$$

Therefore $Re\,(p(z)) = 0$ in A_1

(solution of the Dirichlet problem with zero boundary conditions.)

and $Im\,(p(z)) = C$ on A_1.

But then

$$\ln R_1 \arg F(z) - \ln R_2 \arg z = C$$

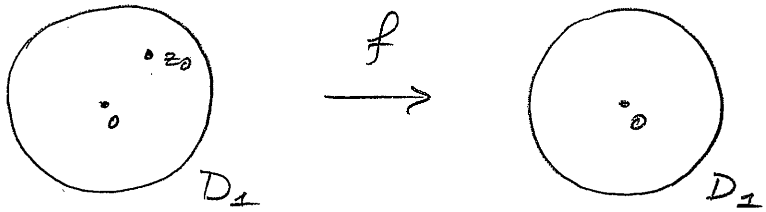
$$\Rightarrow \arg F(z) = \frac{\ln R_2}{\ln R_1} \arg z + C$$

Since any circle C on A_1 is mapped onto a simple contour on A_2

$$2\pi = \Delta_C \arg F(z) = \frac{\ln R_2}{\ln R_1} \Delta_C \arg z = \frac{\ln R_2}{\ln R_1} \cdot 2\pi$$

$$\Rightarrow \frac{\ln R_2}{\ln R_1} = 1 \Rightarrow \ln R_2 = \ln R_1. \Rightarrow R_2 = R_1.$$

#8.



If f is a conformal bijection $D_1 \to D_1$

and $\varphi(z) = \dfrac{z - z_0}{z_0^* z - 1}$

then $g = f \circ \varphi^{-1}$ and $g^{-1} = \varphi \circ f^{-1}$ are both conformal bijections such that $g(0) = 0$

$g^{-1}(0) = 0$.

and $\left.\begin{array}{l} |g(z)| \le 1 \\ |g^{-1}(z)| \le 1 \end{array}\right\}$ for $|z| \le 1$.

By Schwarz's lemma

$$|g(z)| \le |z| \quad \text{and} \quad |g^{-1}(z)| \le |z|$$

$$\Rightarrow \quad |z| = |g^{-1}(g(z))| \le |g(z)|$$

$$\Rightarrow \quad |z| = |g(z)|.$$

$$\Rightarrow \quad g(z) = e^{i\beta(z)} \cdot z$$ — $\beta(z)$ – real-valued, analytic

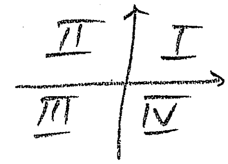
$$\Rightarrow \quad g(z) = e^{i\beta} \cdot z$$ — β is a constant

$$\Rightarrow \quad f \circ \varphi^{-1}(z) = e^{i\beta} \cdot z$$

$$\Rightarrow \quad f(z) = e^{i\beta} \cdot \varphi(z) = e^{i\beta} \frac{z - z_0}{z_0^* z - 1}.$$

($\beta \in \mathbb{R}$ is a constant.)

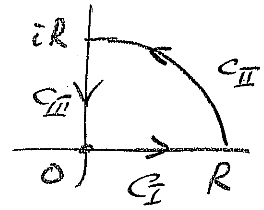
#9. $f(z) = 2z^4 + z^3 + 2z^2 + 1$



$$\#\text{zeros}\,(f \text{ inside } C) = \frac{1}{2\pi}\Delta_C \arg f(z)$$

<u>Quadrant I</u>:

take $C = C_R$ as shown with R large enough.



Then $\Delta_{C_I} \arg f(z) = 0$ since $f(z)$ is real on C_I.

Since $\dfrac{f(z)}{z^4} = 2 + \dfrac{1}{z} + \dfrac{2}{z^2} + \dfrac{1}{z^4} \approx 2$ when $|z| = R$ and R large

$$\Delta_{C_{II}} \arg f(z) = \Delta_{C_{II}} z^4 = 2\pi$$

On C_{III}: $z = iy$, $y > 0$, $f(z) = 2y^4 + 2y^2 + 1 - iy^3$

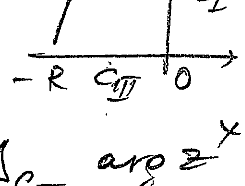
Thus $\operatorname{Re} f(z) > 1$ and therefore

$$\Delta_{C_{III}} \arg f(z) = 0.$$

$\Rightarrow \Delta_{C_R} \arg f(z) = 2\pi \Rightarrow$ one zero in Quadrant I

By symmetry $\left(f(z^*) = f(z)^*\right)$ there must be one zero in Quadrant IV

<u>Quadrant II</u>:



$\Delta_{C_I} \arg f(z) = 0$ as shown previously

$\Delta_{C_{III}} \arg f(z) = 0$ since $f(z)$ is real on $\mathbb{R}$

$$\Delta_{C_{II}} \arg f(z) = \Delta_{C_{II}} \arg z^4$$

$= 2\pi \Rightarrow$ one zero in Q II and by symmetry one in Q III.

#10. Suppose $f(z) \neq 0$ inside C. Then $\frac{1}{f(z)}$ is analytic inside C $\Rightarrow$ $\max \left| \frac{1}{f(z)} \right|$ is achieved on C. Also $f(z)$ is analytic inside $C \Rightarrow \max |f(z)|$ is achieved on C. But $|f(z)|$ is constant on C $\Rightarrow$

$$\max |f(z)| = m_0 = \min |f(z)| = \frac{1}{\max |1/f(z)|}$$

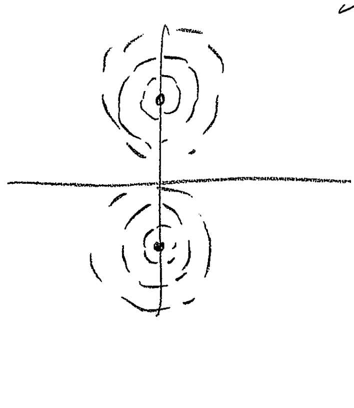
$\Rightarrow$ $|f(z)| = \text{const}$ inside C

$\Rightarrow$ $f(z) = \text{const}$ inside C. — contradiction.

The statement about the zeros of $f'(z)$ cannot possibly be true:

Example $f(z) = z^2+1 = (z+i)(z-i)$;

Contours of constant modulus look as follows (for small enough $|f(z)|$)



However, the only zero of the derivative is at $z=0$, which is on neither of these contours.