May 2, 2019 MATH 655 Prof. V. Panferov

Homework 12

1. (Problem 4-2:1) If the function $f(\zeta)$ is analytic in a domain D' and if the transformation $f : D' \to D$ is one-to-one and onto show that f is conformal on D' (*i. e.* $f'(\zeta) \neq 0$, $\zeta \in D'$).

2. (Problem 4-2:2) If $F(z)$ is analytic inside and on a simple contour C and assumes any value at most once on C show that the transformation $\zeta = F(z)$ maps the domain interior to C conformally and bijectively onto the domain interior to C'.

3. (Problem 4-2:5) Find the critical points of all the transformations defined by the equation $\cos z = \sinh \zeta$.

4. (Problem 4-2:9) (a) Show that the transformation $\zeta = 2z^{-1/2} - 1$ maps the domain exterior to the parabola $y^2 = 4(1-x)$ conformally onto the domain $|\zeta| < 1$. Explain carefully why this transformation does not, at the same time, map the domain interior to the parabola onto the domain $|\zeta| > 1$.

   (b) Show that the transformation $\zeta = \tan^2 \frac{\pi\sqrt{z}}{4}$ maps the domain interior to the parabola $y = 4x(1-x)$ onto the domain $|\zeta| < 1$.

5. Give an example of a conformal map from the semidisk $\{z : |z| < 1, \operatorname{Re}(z) > 0\}$ onto the interior of the unit circle $|z| = 1$.

6. Conformally map the region inside the disk $\{z : |z-1| \leq 1\}$ and outside the disk $\{z : |z - \frac{1}{2}| \leq \frac{1}{2}\}$ onto the upper half-plane.

7. (Problem 4-2:15) A transformation $\zeta = F(z)$ gives a one-to-one conformal map of the domain $1 < |z| < R_1$ onto the domain $1 < |z| < R_2$. If $p(z) = \ln R_1 \ln F(z) - \ln R_2 \ln z$ show that $\operatorname{Im}(p(z))$ is constant in $1 < |z| < R_1$ and hence $R_1 = R_2$.

8. (Problem 4-2:17) Find the form of the most general bijective conformal transformation $\zeta = f(z)$ that maps the interior of the unit circle onto itself, with $f(z_0) = 0$. *Hint: Note that* $w = (z - z_0)/(z_0^* z - 1)$ *is one such transformation, and use Schwarz's lemma given in Exercise 2-6:3 (Problem 6 of Homework Set 8).*

9. (Problem 2-7:9) Use the Argument Principle to determine the quadrants in which the roots of $2z^4 + z^3 + 2z^2 + 1 = 0$ lie.

10. (Problem 2-7:11) Let $f(z)$ be analytic and nonconstant inside a closed contour C on which $|f(z)|$ is constant. Show that $f(z)$ has at least one zero inside C. Is it true that if $f(z)$ has n zeros inside that contour then $f'(z)$ has $n-1$ zeros inside C?