# HOMEWORK 11 – SOLUTIONS



#1. (a) If $P_n = \prod_{j=1}^{n}(1+u_j) \longrightarrow P \neq 0$

then $P_{n-1} \to P \Rightarrow \dfrac{P_n}{P_{n-1}} = 1 + u_n \longrightarrow 1$

$$\Rightarrow u_n \to 0.$$

(b) Suppose $P_n \to P \neq 0$. Then $\dfrac{P_n}{P} \to 1$

$$\Rightarrow \ln_p \frac{P_n}{P} \to 0$$

Also for $n$ large enough,

$$\ln_p \frac{P_{n+1}}{P} - \ln_p \frac{P_n}{P} = \ln_p \frac{P_{n+1}}{P_n}.$$

Then

$$\ln_p \frac{P_n}{P} = \sum_{j=1}^{n} \ln_p (1+u_j) - \ln_p P + 2\pi i k_n$$

$$\Rightarrow 2\pi i \left(k_{n+1} - k_n\right) = \ln_p \frac{P_{n+1}}{P_n} + \ln_p (1+u_{n+1}) \xrightarrow[n\to\infty]{} 0$$

Thus for $n$ large enough $k_{n+1} - k_n = 0$

$$\Rightarrow k_n = \text{const}$$

$$\Rightarrow \sum_{j=1}^{n} \ln_p (1+u_j) = \ln_p \frac{P_n}{P} + \ln_p P - 2\pi i k$$

$$\longrightarrow \ln_p P - 2\pi i k.$$

Conversely, suppose $\sum_{j=1}^{\infty} \ln_p (1+u_j)$ converges.

Then $S_n = \sum_{j=1}^{n} \ln_p (1+u_j)$ converges to $S \in \mathbb{C}$

Then
$$e^{S_n} = e^{\sum_{j=1}^{n} \ln (1+u_j)} = e^{\ln \prod_{j=1}^{n}(1+u_j) + 2\pi i k_n}$$
$$= e^{\ln \prod_{j=1}^{n}(1+u_j)} = \prod_{j=1}^{n}(1+u_j) \longrightarrow e^{S}.$$

Example $\quad u_n = e^{\pi i a b^n} - 1 \; ; \quad a = \frac{1}{8} ; \; b = 0.9$ 

$$1 + u_n = e^{\pi i a b^n}$$

$$\operatorname{lnp}(1+u_n) = \pi i a b^n$$

$$\sum_{n=1}^{\infty} \operatorname{lnp}(1+u_n) = \pi i \frac{ab}{1-b} = \frac{9\pi i}{8}$$

$$\prod_{i=1}^{\infty} e^{\pi i a b^n} = e^{\frac{9\pi i}{8}}$$

$$\operatorname{lnp}\left(\prod_{i=1}^{\infty} e^{\pi i a b^n}\right) = \operatorname{lnp}\left(\frac{9\pi i}{8}\right) = -\frac{7\pi i}{8}$$

(c) $\quad \prod_{n=1}^{\infty}(1+|u_n|)$ converges $\Longleftrightarrow \sum_{n=1}^{\infty} \operatorname{lnp}(1+|u_n|)$ converges

by part (b).

Since $u_n \longrightarrow 0$

$$\lim_{n\to\infty} \frac{\operatorname{lnp}(1+|u_n|)}{|\operatorname{lnp}(1+u_n)|} = \lim_{n\to\infty} \frac{|u_n| - \frac{1}{2}|u_n|^2 + \dots}{|u_n - \frac{1}{2}u_n^2 + \dots|} = 1$$

and

$$\lim_{n\to\infty} \frac{\operatorname{lnp}(1+|u_n|)}{|u_n|} = \lim_{x\to 0+} \frac{\ln(1+x)}{x} = \lim_{x\to 0+} \frac{1}{1+x} = 1$$

(indices $n$ when $u_n = 0$ can be omitted since they do not affect the value of the product or any of the sums.)

By the limit comparison test series (2) and (3) as well as (2) and (4) are either both convergent or both divergent

$\Rightarrow$ same applies to series (3) and (4).

We could also use explicit comparison:

$$|u_n| - \frac{1}{2}|u_n|^2 \leq \log(1+|u_n|) \leq |u_n| \ ,$$ by alternating series

For $n$ large enough $|u_n| \leq 1 \Rightarrow$

$$|u_n| - \frac{1}{2}|u_n|^2 \geq \frac{1}{2}|u_n|$$

$\Rightarrow$ (2) and (4) are either both convergent or both divergent.

Similarly,

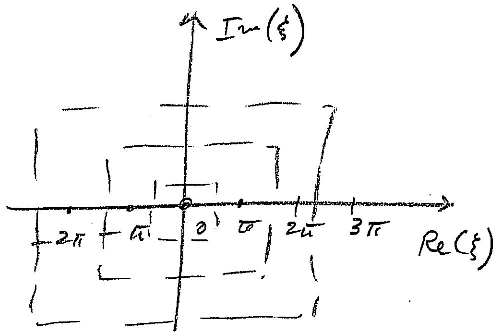
$$\left|u_n - \frac{1}{2}u_n^2\right| \leq |\log(1+u_n)| \leq |u_n|$$

so the same principle applies to the pair of series (3) and (4).

**#2.**

$$I_n = \frac{1}{2\pi i}\int_{C_n} \frac{d\xi}{(\xi - z)^2 \sin\xi}$$

$C_n$ - square with corners $\pi\left(n+\frac{1}{2}\right)(\pm 1, \pm i)$.




$$0 = \lim_{n\to\infty} I_n = \sum \operatorname{Res} \frac{1}{(\xi - z)^2 \sin\xi}$$

$$= \operatorname{Res}_{\xi = z} + \operatorname{Res}_{\xi = 0} + \sum_{n\neq 0} \operatorname{Res}_{\xi = \pi n}$$

$$\operatorname{Res}_{\xi = z} = \left(\frac{1}{\sin\xi}\right)'\Bigg|_{\xi = z} = \frac{-\cos z}{\sin^2 z}$$

$$\operatorname{Res}_{\xi = 0} = \frac{1}{z^2}$$

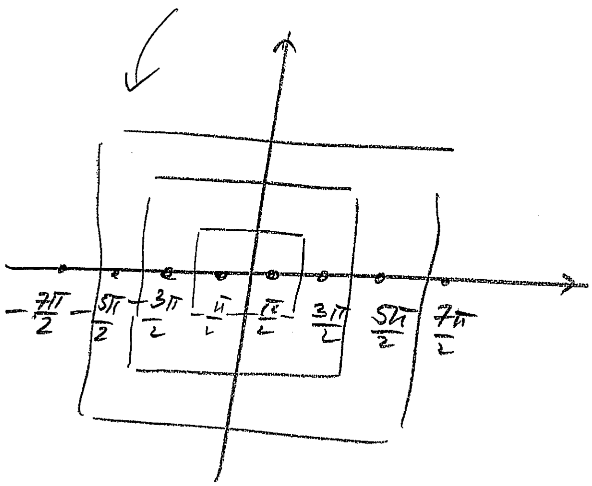
$$\operatorname{Res}_{\xi = \pi n} = \frac{(-1)^n}{(\pi n - z)^2}$$

$$\Rightarrow \quad \frac{\cos z}{\sin^2 z} = \csc z \cot z = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(\pi n - z)^2}$$

#3 (a)

$$I_n = \frac{1}{2\pi i} \int_{C_n} \frac{d\xi}{\xi(\xi - z)\cos\xi}$$

$C_n$ - square with corners $\pi n(\pm 1, \pm i)$




$$|\cos\xi|^2 = |\cos(x+iy)|^2$$
$$= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$$
$$= \cos^2 x(1+\sinh^2 y) + \sin^2 x \sinh^2 y$$
$$= \cos^2 x + \sinh^2 y \geq 1 \text{ on } C_n.$$

Since $\xi(\xi - z) \sim n^2$, length$(C_n) \sim n$

$$\Rightarrow \lim_{n\to\infty} I_n = 0$$

$$\Rightarrow 0 = \lim_{n\to\infty} I_n = \operatorname{Res}_{\xi=0} + \operatorname{Res}_{\xi=z} + \sum_{n=-\infty}^{\infty} \operatorname{Res}_{\xi=\frac{\pi}{2}+\pi n}$$

$$\operatorname{Res}_{\xi=0} = -\frac{1}{z}; \quad \operatorname{Res}_{\xi=z} = \frac{1}{z\cos z}$$

$$\operatorname{Res}_{\xi=\frac{\pi}{2}+\pi n} = \frac{(-1)^{n-1}}{\left(\frac{\pi}{2}+\pi n\right)\left(\frac{\pi}{2}+\pi n - z\right)}$$

since $\frac{1}{\cos z} = -\frac{1}{\sin(z-\frac{\pi}{2})} = -\frac{1}{z-\frac{\pi}{2}} + \dots$

$\frac{1}{\cos z} = \frac{1}{\sin(z+\frac{\pi}{2})} = \frac{1}{z+\frac{\pi}{2}} + \dots$ etc.

Thus
$$\frac{1}{z\cos z} = \frac{1}{z} - \sum_{n=-\infty}^{\infty} \frac{(-1)^{n-1}}{\left(\frac{\pi}{2}+\pi n\right)\left(\frac{\pi}{2}+\pi n - z\right)}$$

Since

$$\frac{z}{\frac{\pi}{2}+\pi n-z} = \frac{\left(z-\left(\frac{\pi}{2}+\pi n\right)\right)+\left(\frac{\pi}{2}+\pi n\right)}{\frac{\pi}{2}+\pi n-z} = -1+\frac{\frac{\pi}{2}+\pi n}{\frac{\pi}{2}+\pi n-z},$$

$$\frac{1}{\cos z} = 1+\sum_{n=-\infty}^{\infty}(-1)^n\left(-\frac{1}{\frac{\pi}{2}+\pi n}+\frac{1}{\frac{\pi}{2}+\pi n-z}\right)$$

$$= 1-2\sum_{n=0}^{\infty}(-1)^n\frac{1}{\frac{\pi}{2}+\pi n}+\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{\frac{\pi}{2}+\pi n-z}+\frac{1}{\frac{\pi}{2}+\pi n+z}\right)$$

$$= 1-\frac{4}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}+\sum_{n=0}^{\infty}(-1)^n\frac{2\left(\frac{\pi}{2}+\pi n\right)}{\left(\frac{\pi}{2}+\pi n\right)^2-z^2},$$

Since $\frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}+\dots = \sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}$,

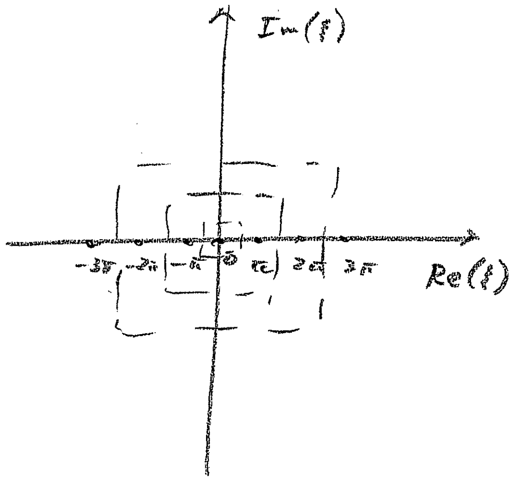
$$\frac{1}{\cos z} = 2\pi\sum_{n=0}^{\infty}(-1)^n\frac{\left(n+\frac{1}{2}\right)}{\pi^2\left(n+\frac{1}{2}\right)^2-z^2}.$$

(c) (part (b) on the next page)



$$I_n = \frac{1}{2\pi i}\int_{C_n} \frac{\cot\xi \, d\xi}{\xi(\xi - z)}$$

$C_n$ – square with the corners $\pi(n+\frac{1}{2})(\pm 1, \pm i)$




$$|\cot\xi|^2 = \frac{|\cos\xi|^2}{|\sin\xi|^2} = \frac{\cos^2 x + \sinh^2 y}{\sin^2 x + \sinh^2 y}$$

on vertical sides of $C_n$,

$$\cos^2 x = 0, \quad \sin^2 x = 1$$

$$\Rightarrow |\cot\xi|^2 \le 1$$

on horizontal sides of $C_n$,

$$|\cot\xi|^2 = \frac{1 + \cos^2 x/\sinh^2 y}{1 + \sin^2 x/\sinh^2 y} \le 1 + \varepsilon$$

if $n$ is large enough.

on $C_n$, $\xi(\xi - z) \sim n^2$; length$(C_n) \sim n$

$$\Rightarrow \lim_{n\to\infty} I_n = 0$$

$$0 = \lim_{n\to\infty} I_n = \mathrm{Res}_{\xi=0} + \mathrm{Res}_{\xi=z} + \sum_{n\neq 0} \mathrm{Res}_{\xi=\pi n}$$

$$\mathrm{Res}_{\xi=0} = -\frac{1}{z^2}, \quad \mathrm{Res}_{\xi=z} = \frac{1}{z}\cot z$$

$$\mathrm{Res}_{\xi=\pi n} = \frac{(-1)^n}{\pi n(\pi n - z)(-1)^n} = \frac{1}{\pi n(\pi n - z)}$$

$$\frac{1}{z}\cot z = \frac{1}{z^2} - \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{1}{\pi n(\pi n - z)} = \frac{1}{z^2} - \sum_{n=1}^{\infty}\frac{1}{\pi n(\pi n - z)} - \sum_{n=1}^{\infty}\frac{1}{(-\pi n)(-\pi n - z)}$$

$$= \frac{1}{z^2} + \sum_{n=1}^{\infty}\left(\frac{1}{\pi n(z - \pi n)} - \frac{1}{\pi n(z + \pi n)}\right)$$

$$= \frac{1}{z^2} + \sum_{n=1}^{\infty}\frac{2\pi n}{\pi n(z^2 - \pi^2 n^2)} = \frac{1}{z^2} + \sum_{n=1}^{\infty}\frac{2}{z^2 - \pi^2 n^2}.$$

$$\Rightarrow \quad \cot z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - \pi^2 n^2}.$$

(b) We can obtain (b) from (c) by shifting by $\frac{\pi}{2}$:

$$\tan z = -\cot\left(z - \frac{\pi}{2}\right)$$

$$\cot z = \sum_{n=-\infty}^{\infty} \frac{1}{z - \pi n} \Rightarrow \cot\left(z - \frac{\pi}{2}\right) = \sum_{n=-\infty}^{\infty} \frac{1}{z - \pi(n+\frac{1}{2})}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z - \pi(n+\frac{1}{2})} + \sum_{n=0}^{\infty} \frac{1}{z + \pi(n+\frac{1}{2})}$$

$$= \sum_{n=0}^{\infty} \frac{2z}{z^2 - \pi^2(n+\frac{1}{2})^2}$$

$$\Rightarrow \tan z = -\cot\left(z - \frac{\pi}{2}\right) = \sum_{n=0}^{\infty} \frac{2z}{\pi^2(n+\frac{1}{2})^2 - z^2}.$$

<u>Note</u>: Same technique works for obtaining the decomposition of $\frac{1}{\cos z}$ from the one for $\frac{1}{\sin z}$ since

$$\frac{1}{\cos z} = -\frac{1}{\sin(z - \frac{\pi}{2})}.$$

**#4.** (a) $f(z) = \cos z$

$$\left(\ln f(z)\right)' = \frac{f'(z)}{f(z)} = \frac{-\sin z}{\cos z} = -\tan z = -2z\sum_{n=0}^{\infty}\frac{1}{(n+\frac{1}{2})^2\pi^2 - z^2}$$

(problem 3) $$= \sum_{n=-\infty}^{\infty}\frac{1}{z-\pi(n+\frac{1}{2})}$$

$$\ln f(z) - \ln f(0) = \sum_{n=-\infty}^{\infty} \ln\left(z-\pi(n+\tfrac{1}{2})\right) - \ln\left(-\pi(n+\tfrac{1}{2})\right)$$

$$\ln f(z) = \sum_{n=-\infty}^{\infty} \ln\frac{z-\pi(n+\frac{1}{2})}{-\pi(n+\frac{1}{2})} + 2\pi i k_n$$

$$f(z) = e^{\sum_{n=-\infty}^{\infty}\ln\left(1-\frac{z}{\pi(n+\frac{1}{2})}\right)}$$

$$f(z) = \prod_{n=-\infty}^{\infty}\left(1-\frac{z}{\pi(n+\frac{1}{2})}\right) = \prod_{n=0}^{\infty}\left(1-\frac{z}{\pi(n+\frac{1}{2})}\right)\left(1+\frac{z}{\pi(n+\frac{1}{2})}\right)$$

$$= \prod_{n=0}^{\infty}\left(1-\frac{z^2}{\pi^2(n+\frac{1}{2})^2}\right).$$

(b) $$f(z) = \cos z - \sin z = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos z - \frac{1}{\sqrt{2}}\sin z\right)$$

$$= \sqrt{2}\cos\left(\frac{\pi}{4}+z\right).$$

$$\Rightarrow f(z) = \sqrt{2}\prod_{n=-\infty}^{\infty}\left(1-\frac{z+\frac{\pi}{4}}{\pi(n+\frac{1}{2})}\right)$$

$$= \sqrt{2}\underbrace{\prod_{n=-\infty}^{\infty}\left(1-\frac{\pi/4}{\pi(n+\frac{1}{2})}\right)}_{\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \text{ by part (a)}}\left(1-\frac{z}{\pi(n+\frac{1}{2})\left(1-\frac{\pi/4}{\pi(n+\frac{1}{2})}\right)}\right)$$

$$= \prod_{n=-\infty}^{\infty}\left(1-\frac{z}{\pi\left(n+\frac{1}{2}\right)-\frac{\pi}{4}}\right) = \prod_{n=-\infty}^{\infty}\left(1-\frac{z}{\pi\left(n+\frac{1}{4}\right)}\right)$$

$$= \prod_{n=0}^{\infty}\left(1-\frac{z}{\pi\left(n+\frac{1}{4}\right)}\right)\prod_{n=1}^{\infty}\left(1-\frac{z}{\pi\left(-n+\frac{1}{4}\right)}\right)$$

$$= \prod_{n=0}^{\infty}\left(1-\frac{z}{\pi\left(n+\frac{1}{4}\right)}\right)\prod_{n=1}^{\infty}\left(1+\frac{z}{\pi\left(n-\frac{1}{2}+\frac{1}{4}\right)}\right)$$

$$= \prod_{k=0}^{\infty}\left(1-\frac{z}{\pi\left(\frac{2k}{2}+\frac{1}{4}\right)}\right)\prod_{k=0}^{\infty}\left(1+\frac{z}{\pi\left(\frac{2k+1}{2}+\frac{1}{4}\right)}\right)$$

$$= \prod_{n=0}^{\infty}\left(1-(-1)^n\frac{z}{\pi\left(\frac{n}{2}+\frac{1}{4}\right)}\right)$$

$$= \prod_{n=0}^{\infty}\left(1-(-1)^n\frac{2z}{\pi\left(n+\frac{1}{2}\right)}\right).$$

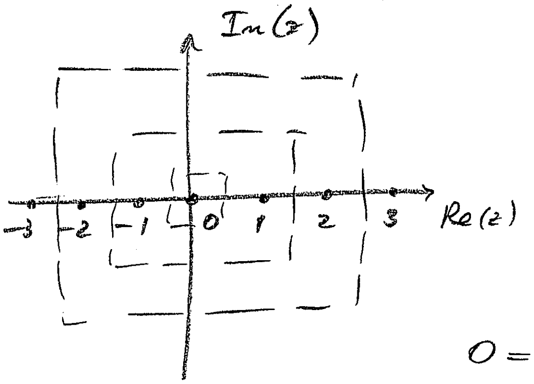
Note: The final form is simply a rearrangement of terms in the product

$$\prod_{n=-\infty}^{\infty}\left(1-\frac{z}{\pi\left(n+\frac{1}{4}\right)}\right) = \underset{(n=0)}{\left(1-\frac{z}{\pi/4}\right)}\underset{(n=-1)}{\left(1+\frac{z}{3\pi/4}\right)}\underset{(n=1)}{\left(1-\frac{z}{5\pi/4}\right)}\cdots$$

#5. Contour integral:

$$I_n = \frac{1}{2\pi i}\int_{C_n} \frac{\pi \cot \pi z}{z^{2p}}\, dz$$

$C_n$ - square with corners $\left(n+\frac{1}{2}\right)(\pm 1, \pm i)$




$|\cot \pi z| \le 1+\varepsilon$ on $C_n$, for $n$ large enough

$\Rightarrow \lim_{n\to\infty} I_n = 0$ for $p \ge 1$.

$$0 = \lim_{n\to\infty} I_n = \operatorname{Res}_{z=0} + \sum_{n\neq 0} \operatorname{Res}_{z=n}$$

$$\cot \pi z = \frac{1}{\pi z}\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!}(\pi z)^{2n}$$

$$\operatorname{Res}_{z=0} \frac{\pi \cot \pi z}{z^{2p}} = (-1)^p \frac{2^{2p} B_{2p}}{(2p)!}\pi^{2p}$$

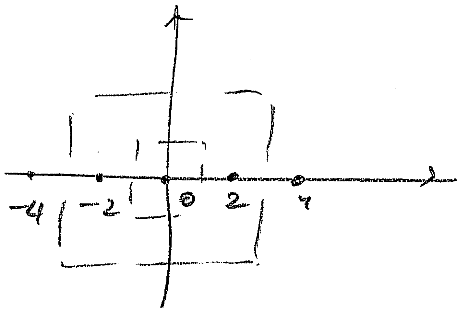
$$\operatorname{Res}_{z=n} \frac{\pi \cot \pi z}{z^{2p}} = \frac{1}{n^{2p}}$$

$$\Rightarrow (-1)^p \frac{(2\pi)^{2p} B_{2p}}{(2p)!} + 2\sum_{n=1}^{\infty} \frac{1}{n^{2p}} = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^{2p}} = (-1)^{p+1} \frac{(2\pi)^{2p} B_{2p}}{2(2p)!}$$

To compute the sum $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2p}}$, try

$$I_n = \frac{1}{2\pi i}\int_{C_{2n}} \frac{\pi/2 \cot \pi z/2}{z^{2p}}\,dz$$




Then $\lim_{n\to\infty} I_n = 0$, so

$$0 = \operatorname{Res}_{z=0} + \sum_{n\neq 0} \operatorname{Res}_{z=2n}$$

$$\Rightarrow 2\sum_{n=1}^{\infty} \frac{1}{(2n)^{2p}} = -\operatorname{Res}_{z=0} \frac{\pi/2 \cot \pi z/2}{z^{2p}}$$

Since $\cot \frac{\pi z}{2} = \frac{2}{\pi z} \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} \left(\frac{\pi z}{2}\right)^{2n}$

$$\Rightarrow \operatorname{Res}_{z=0} \frac{\pi/2 \cot \pi z/2}{z^{2p}} = (-1)^p \frac{\pi^{2p} B_{2p}}{(2p)!}$$

Then $$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2p}} = \sum_{n=0}^{\infty} \frac{1}{n^{2p}} - \sum_{n=0}^{\infty} \frac{1}{(2n)^{2p}}$$

$$= (-1)^{p+1} \frac{(2^{2p}-1)\pi^{2p} B_{2p}}{2(2p)!}$$

#6

(a) $$\csc z = \frac{1}{z} - 2z\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2 - z^2}$$

$$= \frac{1}{z} - \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n\pi - z} - \frac{1}{n\pi + z}\right)$$

$$= \sum_{n=-\infty}^{\infty}(-1)^{n+1}\frac{1}{n\pi - z}$$

$$\Rightarrow \csc\frac{\pi}{2} = 1 = \sum_{n=-\infty}^{\infty}(-1)^{n+1}\frac{1}{n\pi - \pi/2}$$

$$\Rightarrow \frac{\pi}{2} = \sum_{n=-\infty}^{\infty}(-1)^{n+1}\frac{1}{2n-1}$$

$$= 2 - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \cdots$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$\Rightarrow \frac{\pi}{8} = \frac{1}{2}\left(\left(1-\frac{1}{3}\right) + \left(\frac{1}{5}-\frac{1}{7}\right) + \left(\frac{1}{9}-\frac{1}{11}\right) + \cdots\right)$$

$$= \frac{1}{3\cdot 1} + \frac{1}{5\cdot 7} + \frac{1}{9\cdot 11} + \cdots$$

(b) $$\csc z = \frac{1}{z} - 2z\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\pi^2 - z^2}$$

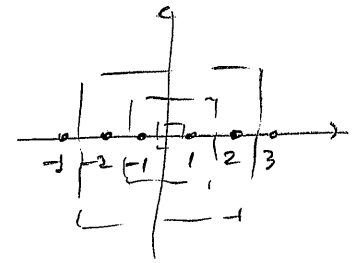
$$\csc\frac{\pi}{2} = 1 = \frac{2}{\pi} - 2\cdot\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\pi^2 - \pi^2/4}$$

$$\Rightarrow \frac{\pi}{4} - \frac{1}{2} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{4n^2-1}$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(2n-1)(2n+1)}$$

$$= \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots$$

(c) Prove that $\frac{7\pi^4}{720} = 1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \cdots$




Compute the contour integral:

$$I_n \; \frac{1}{2\pi i} \int_{C_n} \frac{\pi}{z^4 \sin(\pi z)} \, dz .$$

where $C_n$ is the square with corners $(n+\frac{1}{2})(\pm 1 \pm i)$.

As previously shown, $\lim_{n\to\infty} I_n = 0 \Rightarrow$

$$0 = \lim_{n\to\infty} I_n = \operatorname{Res}_{z=0} \frac{\pi}{z^4(\sin \pi z)} + \sum_{n\neq 0} \operatorname{Res}_{z=n} \frac{\pi}{z^4(\sin \pi z)}$$

Since $\frac{1}{\sin z} = \frac{1}{z} + \frac{1}{6} z + \frac{7}{360} z^3 + \cdots,$

$$\frac{\pi}{z^4 \sin \pi z} = \frac{1}{z^5} + \frac{\pi^2}{6} \frac{1}{z^3} + \frac{7\pi^4}{360} \frac{1}{z} + \cdots,$$

$$\operatorname{Res}_{z=0} \frac{\pi}{z^4 \sin \pi z} = \frac{7\pi^4}{360}$$

$$\operatorname{Res}_{z=n} \frac{\pi}{z^4 \sin \pi z} = \frac{(-1)^n}{n^4}$$

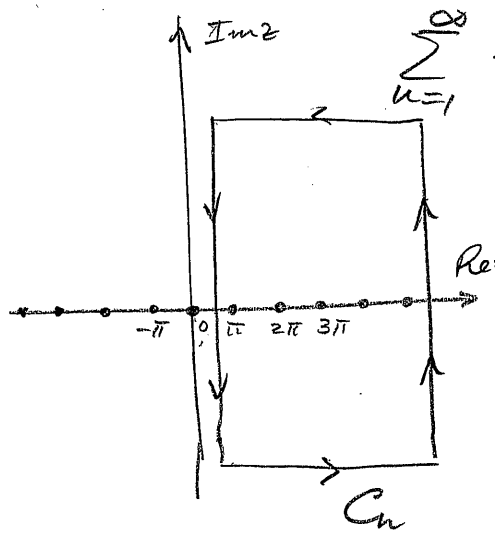
$$\Rightarrow \frac{7\pi^4}{360} = -\sum_{n\neq 0} \operatorname{Res}_{z=n} \frac{\pi}{z^4 \sin \pi z} =$$

$$= 2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^4}$$

$$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^4} = \frac{7\pi^4}{720}.$$

#7. Integrate $(\pi \cot \pi z)/z^3$ to show

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \int_0^{\infty} \frac{(\pi/4)\,\mathrm{sech}^2 \pi y - y \tanh \pi y}{\left(\frac{1}{4}+y^2\right)^2}\, dy.$$




Choose as $C_n$ the rectangle with corners $\left(\frac{1}{2}, \pm n\right), \left(n+\frac{1}{2}, \pm n\right)$

$$|\cot \pi z|^2 = \frac{|\cos \pi z|^2}{|\sin \pi z|^2} = \frac{\cos^2 \pi x + \sinh^2 \pi y}{\sin^2 \pi x + \sinh^2 \pi y}$$

$\leq 1$ on "$|\;|$"; $\leq 1 + \frac{1}{\sinh^2 \pi y}$

$\leq 1 + \varepsilon$ on "$=$".

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

$\parallel$

$$\Rightarrow \sum_{n=1}^{\infty} \mathrm{Res}_{z=\pi n} \frac{\pi \cot \pi z}{z^3} = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{C_n} \frac{\pi \cot \pi z}{z^3}\, dz$$

$$(\text{opposite orient.}) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\pi \cot\left(\frac{\pi}{2} + iy\pi\right)}{\left(\frac{1}{2}+iy\right)^3}\, d\,iy$$

$$= -\frac{1}{2\pi i} \int_{-\infty}^{\infty} -\frac{\pi \tan(\pi i y)}{\left(\frac{1}{2}+y\right)^3}\, dy = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\left(\frac{1}{2}-iy\right)^3}{\left(\frac{1}{4}+y^2\right)^3}\, \pi i \tanh(\pi y)\, dy$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cancel{\frac{1}{8}} + 3\left(\frac{1}{2}\right)^2(-iy) + 3\left(\frac{1}{2}\right)\cancel{(-iy)^2} + (-iy)^3}{\left(\frac{1}{4}+y^2\right)^3}\, i \tanh(\pi y)\, dy$$

$$= \int_0^{\infty} \frac{\frac{3}{4}y - y^3}{\left(\frac{1}{4}+y^2\right)^3} \tanh(\pi y)\, dy = \int_0^{\infty} \left( \frac{y}{\left(\frac{1}{4}+y^2\right)^3} - \frac{y\left(\frac{1}{4}+y^2\right)}{\left(\frac{1}{4}+y^4\right)^3} \right)$$

$$\tanh(\pi y)\, dy$$

$$= \int_0^\infty \left(\frac{-1/4}{(\frac{1}{4}+y^2)^2}\right)' \tanh(\pi y)\, dy - \int_0^\infty \frac{y}{(\frac{1}{4}+y^2)^2} \tanh(\pi y)\, dy$$

$$= \int_0^\infty \frac{1/4}{(\frac{1}{4}+y^2)^2}\, \pi \operatorname{sech}^2(\pi y)\, dy - \int_0^\infty \frac{y}{(\frac{1}{4}+y^2)^2} \tanh(\pi y)\, dy$$

$$= \int_0^\infty \frac{\pi/4 \operatorname{sech}^2(\pi y) - y \tanh(\pi y)}{(\frac{1}{4}+y^2)^2}\, dy .$$