Homework 11

Due on Wed. May 1, 2019.

1. (See p. 18 in the textbook.) Consider an infinite product of complex numbers

$$\prod_{n=1}^{\infty}(1+u_n), \quad \text{with} \quad u_n \neq -1$$

The product is said to *converge* if the limit of partial products $P_n = \prod_{j=1}^{n}(1+u_j)$ exists and is nonzero. Show the following:

(a) If the infinite product converges then $\lim u_n = 0$.

(b) The infinite product converges if and only if the series $\sum_{n=1}^{\infty} \ln_p(1+u_n)$ converges ($\ln_p$ denotes the principal branch of the logarithm), and in that case

$$\lim_{n\to\infty} P_n = \exp\Big(\sum_{n=1}^{\infty} \ln_p(1+u_n)\Big)$$

Give an example when the sum of the series is not equal to $\ln_p(\lim P_n)$.

(c) The product is said to converge absolutely if $\prod_{n=1}^{\infty}(1+|u_n|)$ converges. Show that the convergence of any one of the following expressions implies that of the others:

$$\prod(1+|u_n|) \qquad \sum \ln_p(1+|u_n|) \qquad \sum |\ln_p(1+u_n)| \qquad \sum |u_n|.$$

2. (Problem 3-2:1) Replace the denominator of the integrand in (3-55) by $(\xi - z)^2 \sin\xi$ so as to obtain an expansion for $(\cot z)(\csc z)$.

3. (Problem 3-2:2) Show that

(a) $\dfrac{1}{\cos z} = 2\pi \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(n+\frac{1}{2})}{(n+\frac{1}{2})^2\pi^2 - z^2}$

(b) $\tan z = 2z \displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})^2\pi^2 - z^2}$

(c) $\cot z = \dfrac{1}{z} + 2z \displaystyle\sum_{n=1}^{\infty} \frac{1}{z^2 - n^2\pi^2}$

4. (Problem 3-2:4) Show that

(a) $\cos z = \displaystyle\prod_{n=0}^{\infty}\Big(1 - \frac{z^2}{(n+\frac{1}{2})^2\pi^2}\Big)$

(b) $\cos z - \sin z = \prod_{n=0}^{\infty} \left(1 - \frac{2z(-1)^n}{(n+\frac{1}{2})\pi}\right).$

5. (Problem 3-2:6) In terms of Bernoulli numbers B_n show that, for any $p \in \mathbb{N}$,

$$\sum_{n=1}^{\infty} \frac{1}{n^{2p}} = (-1)^{p+1} \frac{(2\pi)^{2p} B_{2p}}{2(2p)!}$$

and obtain from this the value of the sum

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2p}}.$$

6. (Problem 3-2:8) Prove that

$$\begin{aligned} \frac{\pi}{8} &= \frac{1}{1\cdot 3} + \frac{1}{5\cdot 7} + \frac{1}{9\cdot 11} + \dots \\ \frac{\pi}{4} - \frac{1}{2} &= \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \dots \\ \frac{7\pi^4}{720} &= 1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \dots \end{aligned}$$

7. (Problem 3-2:9) By integrating $(\pi \cot \pi z)/z^3$ over an appropritate contour in the right half-plane, show that

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \int_0^{\infty} \frac{(\pi/4)\,\mathrm{sech}^2 \pi y - y \tanh \pi y}{(\frac{1}{4} + y^2)^2}\, dy.$$

8*. Verify the calculation to show formula (3-58) in the textbook.

9*. Suppose $f(z)$ is a meromorphic function such that each pole z_j is simple, $p_j(z) = a_j z$ and for some integer m the series

$$\sum_{j=1}^{\infty} \frac{|a_j|}{|z_j|^m}$$

converges. Show that the polynomials $g_j(z)$ in formula (2-37) may be taken to be of degree $m-2$.

10*. (Problem 3-1:10) If α and β are real and positive, show that

$$\int_0^{\infty} \frac{\cos \alpha x}{x+\beta}\, dx = \int_0^{\infty} \frac{x e^{-\alpha\beta x}}{1+x^2}\, dx.$$

Which of these two would be easier to compute numerically?