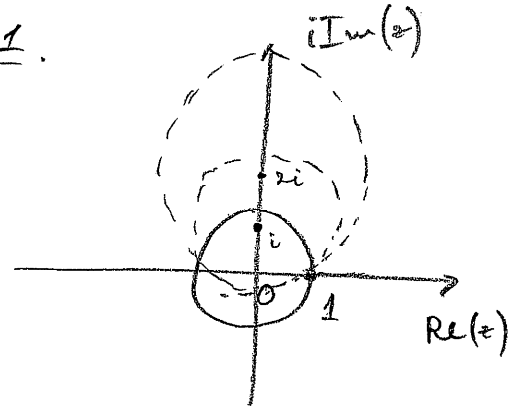
MATH 655 HOMEWORK 10 - SOLUTIONS

#1.



$$f(z) = 1 + z + z^2 + \dots = \frac{1}{1-z}$$

Take $z_0 = i(1-\eta)$, $\eta > 0$

$$f(z) = \frac{1}{1 - z_0 - (z - z_0)} = \frac{1}{1-z_0}\,\frac{1}{1 - \frac{z-z_0}{1-z_0}}$$

$$= \sum_{k=0}^{\infty} (1-z_0)^{-(k+1)} (z-z_0)^k$$

converges for $|z - z_0| < |1 - z_0|$
$= \sqrt{2 - 2\eta + \eta^2}$

Use $z_0 = 2i$

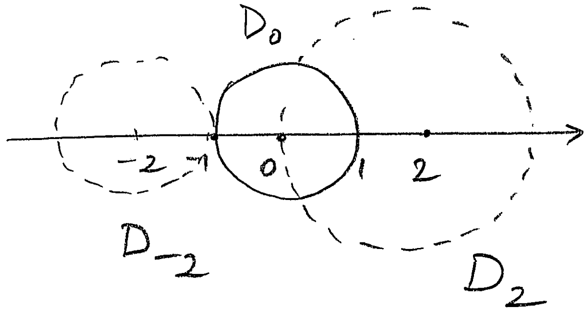
$$f(z) = \frac{1}{1 - 2i - (z - 2i)} = \sum_{k=0}^{\infty} (1-2i)^{-(k+1)} (z-2i)^k$$

converges for $|z - 2i| < |1 - 2i| = \sqrt{5}$

Since $|\pi i - 2i| = |\pi - 2| < 2 < \sqrt{5}$,
the last series also converges
for $z = \pi i$.

#2.

$$f(z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 + \dots$$
$$= \ln(1+z), \quad |z| < 1.$$



Since $D_0 \cap D_2 \neq \emptyset$,
the analytic continuation to D_2
must correspond to the same branch of $\ln(1+z)$:

$$f(z) = \ln\left(3 + (z-2)\right) = \ln(3) + \ln\left(1 + \frac{z-2}{3}\right)$$

$$= \ln 3 + \frac{z-2}{3} - \frac{1}{2}\left(\frac{z-2}{3}\right)^2 + \frac{1}{3}\left(\frac{z-2}{3}\right)^3 - \ldots$$

(converges for $|z-2| < 3 \Rightarrow$ on all of D_2

For $z \in D_{-2}$

$$f(z) = \ln(1+z) = \ln\left(-1 + (z+2)\right) = \ln(-1) + \ln\left(1-(z+2)\right)$$

$$= \pi i + 2\pi i n + (z+2) - \frac{1}{2}(z+2)^2 + \frac{1}{3}(z+2)^3 - \ldots$$

Continuation with a chain of circles in the upper half-plane gives

$$f(-2) = \pi i \quad (n=0)$$

$$\Rightarrow \quad f(z) = \pi i + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{z^k}{k}$$

Continuation through the lower half-plane gives

$$f(-2) = -\pi i \quad (n=-1)$$

$$\Rightarrow \quad f(z) = -\pi i + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{z^k}{k}$$

These are the only two possibilities without allowing multi-valued continuation,

#3. (a) Show $\frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{i\theta})|^2 \, d\theta$

$$= \sum_{n=0}^{\infty} |a_n|^2 r^{2n}, \quad 0<r<R.$$

$$z = z_0 + re^{i\theta} \Rightarrow \begin{cases} f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n = \sum_{n=0}^{\infty} a_n r^n e^{in\theta} \\ (f(z))^* = \sum_{n=0}^{\infty} a_n^* ((z-z_0)^*)^n = \sum_{n=0}^{\infty} a_n^* r^n e^{-in\theta} \end{cases}$$

$$\Rightarrow \quad |f(z_0+re^{i\theta})|^2 = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n a_m^* r^{n+m} e^{i(n-m)\theta}$$

$$= \sum_{k=0}^{\infty} r^k \sum_{m=0}^{k} a_{k-m} a_m^* e^{i(k-2m)\theta}$$

Since $\int_0^{2\pi} e^{i(n-m)\theta} \, d\theta = \begin{cases} 0, & n \neq m \\ 2\pi, & n = m \end{cases}$

after integration over $(0, 2\pi)$ only the terms with $2m = k$ remain:

$$\int_0^{2\pi} |f(z_0+re^{i\theta})|^2 \, d\theta = \sum_{j=0}^{\infty} r^{2j} |a_j|^2 \cdot 2\pi$$

(b) Let $F(r) = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$.

$|f(z)| \le M$, $|z|<r \Rightarrow F(r) \le \frac{1}{2\pi}\int_0^{2\pi} |f|^2 \, d\theta \le M^2$.

$F(r)$ bounded and increasing on $(0, R)$

$\Rightarrow \lim_{r\to R} F(r)$ exists and is $\le M^2$.

Let $L = \lim_{r \to R} F(r)$.

For each N fixed, $\forall \varepsilon > 0$

$$\sum_{n=0}^{N} |a_n|^2 R^{2n} \le L + \varepsilon$$

$$\Rightarrow \sum_{n=0}^{N} |a_n|^2 R^{2n} \le L$$

$$\Rightarrow \lim_{N \to \infty} \sum_{n=0}^{N} |a_n|^2 R^{2n} \le L$$

Also since $|a_n|^2 R^{2n} \ge |a_n|^2 r^{2n}$

$$\sum_{n=0}^{\infty} |a_n|^2 R^{2n} \ge \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

$$\Rightarrow \sum_{n=0}^{\infty} |a_n|^2 R^{2n} \ge L \Rightarrow \sum_{n=0}^{\infty} |a_n|^2 R^{2n} = L \le M^2$$

(c) $z_0 = 0$, $R = 1$

$$\Rightarrow \sum_{n=0}^{\infty} |a_n|^2 \le M^2$$

$$\Rightarrow |a_n|^2 \xrightarrow[n \to \infty]{} 0 \Rightarrow a_n \xrightarrow[n \to \infty]{} 0.$$

Since the general term of a convergent series must have limit zero as $n \to \infty$.

#4. $f(z)=\sum_{n=0}^{\infty} z^{2^n}$; $z \in D=\{z: |z|<1\}$ (5)

(a) $f(z)=\sum_{n=0}^{\infty} a_n z^n$, $a_n = \begin{cases} 1, & n=2^k \\ 0, & \text{otherwise.} \end{cases}$

$$\frac{1}{R} = \overline{\lim_{n\to\infty}} |a_n|^{\frac{1}{n}} = 1 \Rightarrow R=1.$$

(b) By previous problem, $a_n \not\to 0$, $n\to\infty$

$\Rightarrow$ $f(z)$ is unbounded for $|z|<1$;

since $|f(z)| \le \sum_{n=0}^{\infty} |a_n| |z|^n$

$$\Rightarrow \sup_{|z|=\rho} |f(z)| \le \sum_{n=0}^{\infty} a_n \rho^n = f(\rho)$$

$$\Rightarrow \lim_{\rho\to 1} f(\rho) = \lim_{\rho\to 1} \sup_{|z|=\rho} |f(z)| = +\infty.$$

(c) $f(z^2) = \sum_{n=0}^{\infty} z^{2^{n+1}} = \sum_{n=1}^{\infty} z^{2^n} = \left(\sum_{n=0}^{\infty} z^{2^n}\right) - z$

$= f(z) - z.$

$$\Rightarrow \lim_{\rho\to 1} f(-\rho) = \lim_{\rho\to 1} f((-\rho)^2) + \lim_{\rho\to 1} (-\rho)$$

$= +\infty$

(d) By induction, $f\left(z^{2^n}\right) = f(z) - \sum_{j=0}^{n-1} z^{2^j}$

True for $n=1$ by the previous step.

If true for $n=k$, then

$$f\left(z^{2^{k+1}}\right) = f\left((z^2)^{2^k}\right) = f(z^2) - \sum_{j=0}^{k-1} (z^2)^{2^j}$$

$$= f(z) - z - \sum_{j=1}^{k} z^{2^j} = f(z) - \sum_{j=0}^{k} z^{2^j}.$$

$$\Rightarrow \quad f(z) = f\left(z^{2^n}\right) + \sum_{j=0}^{n-1} z^{2^j}$$

$$\lim_{\rho \to 1} f\left(\rho e^{2\pi i \frac{n}{2^m}}\right) = \lim_{\rho \to 1} f\left(\rho^{2^n}\right) + \underbrace{\lim_{\rho \to 1} \sum_{j=0}^{\infty} \left(\rho e^{2\pi i \frac{n}{m}}\right)^{2^j}}_{\text{finite in } \mathbb{C}}$$

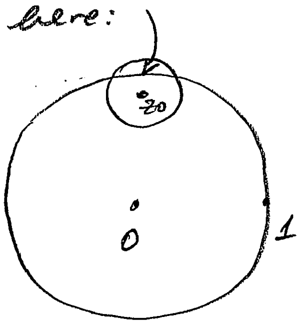
$$= \infty$$

(e) The set $S = \left\{ z_{nm} = e^{2\pi i \frac{n}{2^m}} : n, m \in \mathbb{N} \right\}$

is dense in $\{z : |z| = 1\}$

(For m large enough the distance between z_{nm} and $z_{n+1,m}$ is as small as we want)

Impossible to continue: $f(z)$ is unbounded here:



$\Rightarrow$ A Taylor series centered at $z_0 \in D$ cannot have radius of convergence $> 1 - |z_0|$ since $f(z)$ is unbounded on any finite arc of the unit circle

Therefore, it is impossible to obtain an analytic continuation of $f(z)$ to any region that extends beyond the original disk D.

#5. Suppose

$$u(x,0) = \sin(\pi x); \quad u_y(x,0) = -\pi \sin(\pi x)$$

for $x \in (0,1)$.

If $f(z) = u + iv$; $u_x = v_y$, $u_y = -v_x$

then $f'(z) = u_x + iv_x = u_x - iu_y$.

$$\Rightarrow f(z) = \int_{z_0}^{z} (u_x - iu_y)\,d\xi.$$

Compute
$$p + iq = \int_0^x u_x - iu_y \, dx$$
$$= \int_0^x \pi\cos(\pi x) - i\pi\sin(\pi x)\,dx$$
$$= \sin(\pi x) - i\cos(\pi x)$$

continues analytically to the whole $\mathbb{C}$

as $f(z) = \sin(\pi z) - i\cos(\pi z) = -ie^{i\pi z}$

The values of the real part are determined uniquely by fixing $u(x,0) = \sin(\pi x)$; the values of the imaginary part are determined up to an additive constant:

$$f(z) = -ie^{i\pi z} + i\alpha; \quad \alpha \in \mathbb{R}.$$

#6. Note: Need to assume that Γ is a contour, so integration along Γ is possible.

In Morera's Theorem, it is sufficient to establish the condition

$$\int_C f(z)\,dz = 0$$

for C - rectangles; also for C rectangles $\leq$ some fixed size.

For such a rectangle $C \subseteq U$, we either have $C \subseteq U_1$, or $C \subseteq U_2$ or $C \cap \Gamma \neq \emptyset$.

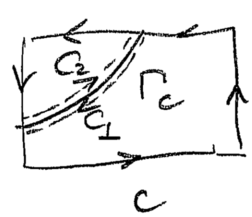
In the two former cases

$$\int_C f(z)\,dz = 0$$

since f is analytic on both U_1 and U_2.

($f(z) = f_1(z)$, $z \in U_1$, $f(z) = f_2(z)$, $z \in U_2$, $f(z)$ defined by continuity on Γ)

If $C \cap \Gamma \neq \emptyset$ we can split



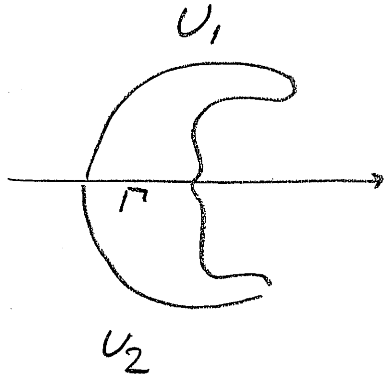
$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{\text{around } \Gamma_C} f(z)\,dz$$

The first two integrals vanish since $C_1 \subseteq U_1$, $C_2 \subseteq U_2$, and the last one vanishes in the limit as C_1, C_2 approach each other.

Therefore $\int_C f(z)\,dz = 0$ for any C in U $\Rightarrow f(z)$ is analytic.

#7: Let $f_1 : U_1 \to \mathbb{C}$ analytic

$U_2 = \{z : z^* \in U_1\}$; $U_1 \cup \Gamma \cup U_2$ - region



Let $f_2 : U_2 \to \mathbb{C}$
$f_2(z) = (f_1(z^*))^*$.

Then f_2 is analytic on U_2.

Indeed, $f_1 = u(x,y) + i\,v(x,y)$
$f_2 = u(x,-y) - i\,v(x,-y)$
$= p(x,y) + i\,q(x,y)$

Then $p_x = u_x = v_y = q_y$

$p_y = -u_y = v_x = -q_x$

$\Rightarrow p, q$ satisfy CR; continuously differentiable in the real sense

$\Rightarrow f_2(z)$ is analytic on U_2.

Further, for $x \in \Gamma$,
$$\lim_{\substack{z \to x \\ (\operatorname{Im} z < 0)}} f_2(z) = \lim_{\substack{z \to x \\ (\operatorname{Im} z < 0)}} (f_1(z^*))^*$$
$$= \Big(\lim_{\substack{z \to x \\ \operatorname{Im} z > 0}} f_1(z)\Big)^*$$

and since the limit is real,

$$\lim_{\substack{z \to x \\ \operatorname{Im} z < 0}} f_2(z) = \lim_{\substack{z \to x \\ \operatorname{Im} z > 0}} f_1(z).$$

Therefore the function

$$f(z) = \begin{cases} f_1(z) & , z \in U_1 \\ \lim\limits_{\substack{\xi \to z \\ Im(\xi) > 0}} f_1(\xi) & , z \in \Gamma \\ f_2(z) & , z \in U_2 \end{cases}$$

is analytic on $U_1 \cup U_2$ and continuous on Γ.

By the previous problem, $f(z)$ is analytic for $z \in U$.

<u>Example</u>: Any fn. $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that $f(z)$ is real when z is real must have $a_n \in \mathbb{R}$

(same proof as for uniqueness of power series...)

Then
$$\left(f(z^*)\right)^* = \left(\sum_{n=0}^{\infty} a_n z^{*n}\right)^*$$
$$= \sum_{n=0}^{\infty} a_n \left((z^*)^n\right)^*$$
$$= \sum_{n=0}^{\infty} a_n \left((z^*)^*\right)^n = \sum_{n=0}^{\infty} a_n z^n$$

<u>Another example</u>:
$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} (1-i)^{-(k+1)} (z-i)^k$$
for $z \in \{z : |z-i| < \sqrt{2}\}$

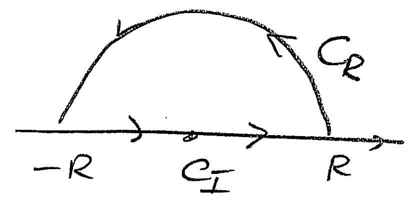
Then $f(z^*)^* = \sum_{n=0}^{\infty} (1+i)^{-(k+1)} (z+i)^k$

provides the series for analytic continuation onto $\{z : |z+i| < \sqrt{2}\}$

$(b^2 < 4ac)$

#8. $$\int_{-\infty}^{\infty} \frac{dx}{ax^2+bx+c} = \int_C \frac{dz}{az^2+bz+c}$$

where $C = C_I + C_R$:



$$\left| \int_{C_R} \frac{dz}{az^2+bz+c} \right| \le \frac{\pi R}{|a|R^2 - |b|R - |c|} \xrightarrow[R\to\infty]{} 0$$

since $|a| > 0$.

$$\int_C \frac{dz}{az^2+bz+c} = 2\pi i \operatorname*{Res}_{z=z_1} \frac{1}{az^2+bz+c}$$

$$\left[\text{where} \quad z_1 = -\frac{b}{2a} + i\,\frac{\sqrt{4ac-b^2}}{2a}\right]$$

$$= 2\pi i \lim_{z\to z_1} \frac{z-z_1}{az^2+bz+c} = 2\pi i \frac{1}{2az_1+b}$$

$$= \frac{2\pi i}{i\sqrt{4ac-b^2}} = \frac{2\pi}{\sqrt{4ac-b^2}}.$$

9.

$$I=\int_0^\infty \frac{x\sin x}{a^2+x^2}\,dx=\frac{1}{2}\int_{-\infty}^{\infty}\frac{x\sin x}{a^2+x^2}\,dx$$

Calculate

$$\int_C \frac{z e^{iz}}{a^2+z^2}\,dz \quad \text{for } C:$$

$$\left|\int_{C_R}\right|=\left|\int_0^\pi \frac{Re^{i\theta}e^{iR\cos\theta}e^{-R\sin\theta}\,Ri\,e^{i\theta}\,d\theta}{a^2+R^2e^{2i\theta}}\right|$$

$$\leq \frac{2\pi R^2}{R^2-a^2}\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta \leq \frac{2\pi R^2}{R^2-a^2}\int_0^{\pi/2} e^{-R\frac{2}{\pi}\theta}\,d\theta$$

$$=\frac{2\pi R^2}{R^2-a^2}\,\frac{\pi}{2R}\left(1-e^{-R}\right)\xrightarrow[R\to\infty]{} 0$$

(Or simply, $G(R)=\dfrac{R}{R^2-a^2}\xrightarrow[R\to\infty]{}0$ is a radial majorant for the integrand.)

Thus,

$$\int_C \frac{z e^{iz}}{a^2+z^2}\,dz=\lim_{R\to\infty}\int_{C_I}\frac{z e^{iz}}{a^2+z^2}\,dz=\int_{-\infty}^{\infty}\frac{x e^{ix}}{a^2+x^2}\,dx$$

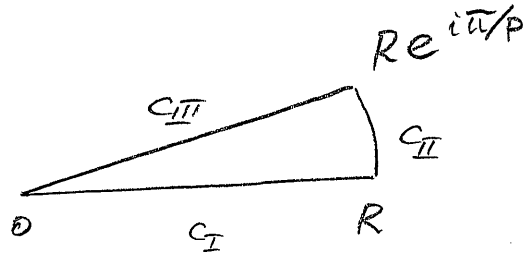
$$\Vert$$

$$2\pi i \operatorname*{Res}_{z=ai}\frac{z e^{iz}}{a^2+z^2}=2\pi i\,\frac{ai\,e^{-a}}{2ai}=\pi i e^{-a}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{x\sin x}{a^2+x^2}\,dx=\pi e^{-a}.$$

\# 10.

$$\int_{-\infty}^{\infty} \frac{x^s\,dx}{1+x^{2p}} = \begin{cases} 0, & s \text{ odd} \\ 2\underbrace{\int_0^{\infty} \frac{x^s\,dx}{1+x^{2p}}}_{I}, & s \text{ even} \end{cases}$$



$$C = C_I + C_{II} + C_{III}$$

$$\int_C \frac{z^s\,dz}{1+z^{2p}} = \int_{C_I} + \int_{C_{II}} + \int_{C_{III}}$$

$$\left|\int_{C_{II}} \frac{z^s\,dz}{1+z^{2p}}\right| \le \frac{\pi}{p}\,\frac{R^{s+1}}{R^{2p}-1} \xrightarrow[R\to\infty]{} 0$$

$$(s+1 < 2p)$$

$$\int_{C_{III}} \frac{z^s\,dz}{1+z^{2p}} = -\int_0^R \frac{e^{i\pi(s+1)/p}\,x^s\,dx}{1+x^{2p}}$$

$$\lim_{R\to\infty} \int_C \frac{z^s\,dz}{1+z^{2p}} = \int_{C_I} + \int_{C_{III}} = \left(1-e^{i\pi(s+1)/p}\right) I$$

$$\|$$

$$2\pi i \operatorname{Res}_{z=e^{i\pi/2p}} \frac{z^s}{1+z^{2p}} = 2\pi i\,\frac{e^{i\pi s/2p}}{2p\,e^{i\pi(2p-1)/2p}}$$

$$= -\frac{2\pi i}{2p}\,e^{i\pi(s+1)/2p}$$

$$I = \frac{2\pi i}{2p} \frac{e^{i\pi(s+1)/2p}}{e^{i\pi(s+1)/p} - 1}$$

$$= \frac{\pi}{2p} \frac{1}{\sin\left(\pi(s+1)/2p\right)}$$

Thus, for s - even

$$\int_{-\infty}^{\infty} \frac{x^s\,dx}{1+x^{2p}} = 2I = \left(1+(-1)^s\right) \frac{\pi/2p}{\sin\left(\frac{\pi}{2p}(s+1)\right)} .$$