**Homework 10**

Due on Fri. Apr 19, 2019.

1. Give an example of a chain of circles that provides an analytic continuation of the function element $f(z) = 1 + z + z^2 + \ldots$ in the disk $|z| < 1$ to $z = \pi i$. Obtain explicitly the Taylor series inside each circle.

2. Show that the function element $(f, D_0)$ given by $f(z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \ldots$ in the unit disk $D_0 = \{z : |z| < 1\}$ has unique analytic continuation to the disk $D_2 = \{z : |z-2| < 2\}$ and two distinct analytic continuations to the disk $D_{-2} = \{z : |z+2| < 1\}$. Obtain the corresponding Taylor series.

3. Suppose $R$ is the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ and let $f(z)$ denote its sum.

   (a) Show that $\displaystyle \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})|^2 \, d\theta = \sum_{n=0}^{\infty} |a_n|^2 \, r^{2n}, \quad 0 < r < R.$
   *Hint:* $|f|^2 = ff^*$. *See also Section 2-6.*

   (b) If $f$ is bounded: $|f(z)| \le M$ for $|z - z_0| < R$, show that $\displaystyle \sum_{n=0}^{\infty} |a_n|^2 \, R^{2n} \le M^2.$

   (c) If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is bounded inside the unit circle $|z| = 1$ show that $a_n \to 0$ as $n \to \infty$.

4. Consider the function element $(f, D)$,

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}, \quad D = \{z : |z| < 1\}.$$

   (a) Show that the series used in the definition of $f(z)$ has radius of convergence 1.

   (b) Show that $f(z)$ is unbounded as $z \to 1$ and moreover that $f(z) \to +\infty$ for $z < 1$ real and approaching 1.

   (c) Show that $f(z^2) = f(z) - z$ for $|z| < 1$. Use this to show that $f(z) \to +\infty$ for $z > -1$ real and approaching $-1$.

   (d) Extend the result of part (c) to show that the radial limits from inside of the unit circle do not exist for any point $z_{nm} = \exp(2\pi i \frac{n}{2^m})$.

   (e) We say that the boundary of a region $U$ is a *natural boundary* for a function $g(z)$ analytic in $U$ if this function has no analytic continuation to any larger region containing $U$. Prove that the function element $(f, D)$ has the unit circle as its natural boundary.

5. Suppose that the real part $u$, and its normal derivative $u_y$, of an analytic function $f(z)$ are given for $x \in (0,1)$:

$$u(x,0) = \sin(\pi x), \quad u_y(x,0) = -\pi \sin(\pi x).$$

Determine the function $f(z)$ on a largest possible domain in $\mathbb{C}$. Is $f(z)$ unique?

*Hint: Use Problem 2-8:1 in the textbook.*

6. (Problem 2-8:2) Let two regions $U_1$ and $U_2$ be adjacent to one another; in more precise terms, for a certain subset $\Gamma$ of their common boundary, the set $U = U_1 \cup \Gamma \cup U_2$ is again a region. Examples could be provided by two rectangles sharing a common side, two half-circles sharing a common diameter, etc. However, two disks sharing a common boundary point are not considered adjacent regions. Let $f_1(z)$ be analytic in $U_1$, $f_2(z)$ in $U_2$; let each function be continuous onto $\Gamma$, and let $f_1(z) = f_2(z)$ on $\Gamma$. Show that the combined function is analytic in $U$.

*Hint: Use Morera's Theorem (Problem 2-3:7). The assumption of simple connectedness is not needed in that statement.*

7. (Problem 2-8:3: Schwarz's Reflection Principle) Let $f(z)$ be analytic in a region $U_1$ in the upper half-plane adjacent to a part $\Gamma$ of the real axis, with $f(z)$ continuous on $\Gamma$ and purely real on $\Gamma$. Let $U_2$ denote the mirror image of $U_1$ in the real axis: $U_2 = \{z : z^* \in U_1\}$ and such that $U = U_1 \cup \Gamma \cup U_2$ is a region in $\mathbb{C}$. Then show that by setting $f(z) = f^*(z^*)$ for $z \in U_2$ we obtain an analytic continuation of $f$ into $U$. Devise a non-trivial example.

8. (Problem 3-1:1) Using the contour of Figure 3-1 in the textbook, show that if $a$, $b$, $c$ are real and $b^2 < 4ac$ then

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2 + bx + c} = \frac{2\pi}{(4ac - b^2)^{1/2}}.$$

9. (Problem 3-1:2) Using the contour of Figure 3-1 in the textbook, evaluate (using Jordan's lemma where necessary)

$$I = \int_0^{\infty} \frac{x \sin x}{a^2 + x^2}\, dx.$$

10. (Problem 3-1:5) If $p$ and $s$ are positive integers with $s \leq 2p - 2$, show that

$$\int_{-\infty}^{\infty} \frac{x^s\, dx}{1 + x^{2p}} = (1 + (-1)^s)\, \frac{\pi/(2p)}{\sin((\pi/(2p))(s+1))}.$$