Midterm 3: Study Guide

Textbook coverage:

3.3: Chain Rule; Implicit Differentiation; Derivatives of Logarithms. Examples 1, 2, 3, 5.

3.4: Derivative Rules for Sine and Cosine; Derivative Rules for Other Trigonometric Functions; Examples 2, 4.

3.5: Linear Approximation; Error Estimates, Sensitivity and Elasticity; Examples 3, 7, 8, 9.

3.6: Higher derivatives; Concave Up/Concave Down; Quadratic Approximation; Examples 6, 7, 8.

4.1: Graphing with Calculus; Examples 1, 2 (problem 15), 3 (problem 6), 4 (problem 31).

4.2: Local Maxima and Minima; Fermat's theorem; Critical Points; First and Second Derivative Tests; Global Extrema; Closed and Open Interval Methods; Examples 3, 4, 6, 9.

4.3: Steps to Solve an Optimization Problem; Examples 3, 4.

4.4: Examples 1, 3.

List of Review Questions

1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the following functions:

   (a) $y = (x+2)e^{3x}$

   (b) $y = \log_2(2x^2+4)$

   (c) $y = e^{\sin(x)}$

   (d) $y = \frac{0.02}{(3x+4)^{100}}$

   (e) $y = 7 - 6\sin(\frac{\pi}{6}(x-2))$

2. Using linear approximation for the function $y = x^{1/3}$ estimate the value $65^{1/3}$ without using a calculator. Based on the second derivative, determine whether the linear approximation overestimates or underestimates the true answer.

3. Find the linear and quadratic approximations to the function $y = e^{-2x}$ at $x = 1$. Graph the function and its approximations over the interval $(0, 2)$.

4. (a) Find the sensitivity of $y = f(x)$ at the point $x = a$ and use it to estimate Δy for the given measurement error Δx:

$$y = \sqrt{2x^2+1}, \quad a = -2, \quad \Delta x = 0.01.$$

(b) Find the elasticity of $y = f(x)$ at the point $x = a$ and use it to estimate the percent error in y for the given percent error in x:

$$y = \sqrt{2x^2 + 1}, \quad a = -2, \quad \text{with } 8\% \text{ error in } x.$$

5. (a) If your measurement of the radius of a circle is accurate to within 3%, approximate how accurately (to the nearest percentage) is your estimate of the area A when the radius is $r = 12$cm? (Use elasticity for the function $A = \pi r^2$ representing the area of the circle.)

   (b) A certain cell is modeled as a sphere. If the formulas $S = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$ are used to compute the surface area and volume of the sphere, respectively, estimate the effect on S and V produced by a 1% increase in the radius r. Use the linear approximation.

6. A drug is injected into a patient's blood stream. The concentration of the drug in the blood stream t hours after the drug is injected is modeled by the formula

$$C(t) = \frac{0.12t}{t^2 + t + 1}$$

   where C is measured in milligrams per cubic centimeter.

   (a) Compute the sensitivity of C at $t = 0.5$ (corresponding to 30 minutes after injection).

   (b) Use your answer to part (a) to estimate the change in concentration over the time period from 30 to 33 minutes after injection.

   (c) Determine the maximum concentration and the time when it occurs.

7. Let f be a function defined by

$$y = x^3 + 35x^2 - 125x - 9375.$$

   Determine where the function is increasing, where it is decreasing, and where the graph is concave up or concave down.

8. Consider the following approximate function for the blood pressure of a patient:

$$P(t) = 100 + 20\cos\left(\frac{\pi(t - 15)}{35}\right) \quad \text{mmHg}$$

   where t is measured in minutes.

(a) Find $P'(20)$ and interpret.

(b) During what periods of the first hour is the blood pressure increasing?

(c) At what time does the blood pressure achieve its minumum?

9. Sketch the graph of a function $y = f(x)$ with the following properties:

   $f'(x) > 0$ when $x < 1$;

   $f'(x) < 0$ when $x > 1$;

   $f''(x) > 0$ when $x < 1$;

   $f''(x) > 0$ when $x > 1$.

   What happens with the derivative of f when $x = 1$?

10. Check problem 22 in Section 4.1 and figure out what's wrong with it. (There is no function fitting the description given in the problem!)

11. Find the global maximum and the global minimum of $f(x) = \sqrt{x}e^{-x/2}$ on $[0, \infty)$.

12. A model using the function

$$C(t) = \frac{k}{b-a}(e^{-at} - e^{-bt})$$

   has been suggested for describing the concentration in the blood of a drug injected into the body intramuscularly. Here a, b and k are positive constants, with $b > a$. At what time does the largest concentration occur? What happens to the concentration as $t \to \infty$?

13. During the winter, a species of bird migrates from the coast of a mainland to an island 500 miles southeast. If the energy the bird requires to fly one mile over the water is 50% more than the amount of energy it requires to fly over the land, determine what path the species should fly to minimize the amount of energy used. (Assume that the coast line runs from north to south.)

14. For a fish swimming at speed v relative to the water, the energy expenditure per unit time is proportional to v^3. It is believed that migrating fish try to minimize the total

energy required to swim a fixed distance. If the fish are swimming against a current u ($u < v$) then the time required to swim a distance L is $L/(v-u)$ and the total energy E required to swim the distance is given by

$$E(v) = av^3 \frac{L}{v-u}$$

where a is a proportionality constant.

(a) Determine the value v that minimizes $E(v)$.

(b) Sketch the graph of $E(v)$.

Note: This result has been verified experimentally: migrating fish swim against a current at a speed 50% greater than the current speed.

For other types of review questions look up problems similar to ones in online homework. Examples: **4.4:** 1, 3, 11, 13; **4.3:** 13-15, 25, 27, 32; **4.2:** 9, 19, 23; **4.1:** 3, 5, 33, 35; **3.6:** 19, 23, 28, 30, 50; **3.5:** 3, 21, 27, 33, 35; **3.4:** 7, 19, 27, 35; **3.3:** 9, 19, 25, 43; **3.2:** 5, 7, 13, 21, 22, 26, 33.

Answers:

1. (a) $(3x+7)e^{3x}$, $(9x+24)e^{3x}$; (b) $\frac{2x}{(\ln 2)(x^2+2)}$, $\frac{-2(x^2-2)}{(\ln 2)(x^2+2)^2}$; hint: $\log_2 x = (\ln x)/(\ln 2)$ (change of base formula); (c) $\cos x e^{\sin x}$; $(\cos^2 x - \sin x)e^{\sin x}$; (d) $-\frac{6}{(3x+4)^{101}}$; $\frac{1818}{(3x+4)^{102}}$; (e) $-\pi \cos(\frac{\pi}{6}(x-2))$; $\frac{\pi^2}{6}\sin(\frac{\pi}{6}(x-2))$.

2. Hint: $x = 64$ is a convenient value for $y = x^{1/3}$ since $64^{1/3} = 4$. Answer: $4 + 1/48 \approx 4.02083$; this is an overestimate. Use the linear approximation $x^{1/3} \approx 4 + \frac{1}{48}(x - 64)$; since the second derivative $(x^{1/3})'' = -\frac{2}{9}x^{-5/3}$ is always negative, the function is concave down and the graph lies *below* the tangent line.

3. linear: $y = e^{-2}(1 - 2(x-1))$; quadratic: $y = e^{-2}(1 - 2(x-1) + 2(x-1)^2)$; the linear approximation is the tangent line to the graph at $x = 1$; the quadratic one is the tangent parabola (follows the graph closer for x near 1).

4. (a) $f'(-2) = -\frac{4}{3}$, $\Delta y \approx -0.0133$; (b) $E = \frac{8}{9}$, 7.1%.

5. (a) Area increases by ≈6% (b) Area increases by ≈ 2%; volume increases by ≈ 3%.

6. (a) $S = C'(0.5) = 0.029388$ (b) linear estimate: 0.0014694 mg/ml (c) the maximum concentration of $C = 0.04$ mg/ml is achieved when $t = 1$ hour.

7. Increases on $(-\infty, -25)$ and on $(\frac{5}{3}, \infty)$; decreases on $(-25, \frac{5}{3})$; concave down on $(-\infty, -\frac{35}{3})$, concave up on $(-\frac{35}{3}, \infty)$.

8. (a) $P'(20) = -\frac{4\pi}{7}\sin\frac{\pi}{7} \approx -0.77891$. The blood presure is decreasing after 20 minutes, at a rate ≈ 0.78 mmHg/min; (b) increasing on $(0, 15)$ and $(50, 60)$; decreasing on $(15, 50)$; (c) the global minumum is reached at $t = 50$.

9. One possible function is $y = -\sqrt{|x-1|}$. The function is necessarily not differentiable at $x = 1$.

11. Global maximum $\frac{1}{\sqrt{e}}$, achieved at $x = 1$; global minimum 0, achieved at $x = 0$.

12. $t = \frac{\ln b - \ln a}{b - a}$. The concentration decays to zero.

13. 37.3 miles along the coast, then straight to the island.

14. $v = 1.5u$.